Boundary layers in reflected heat-kernel Sinkhorn metrics

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Luminary Oddities

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Abstract

As the regularization time h tends to zero, the heat-kernel Sinkhorn tensor converges to the Riemannian tensor of quadratic optimal transport. On manifolds without boundary the leading bias has order h^2 and is governed by Fisher information, and Lavenant, Luckhardt, Mordant, Schmitzer and Tamanini derived the small-regularization limit formally for the quadratic cost on convex domains. Comparing these results raises the question of the rate and boundary correction on a domain with boundary, which we answer for the intrinsic reflecting heat-kernel cost on smooth compact Riemannian manifolds with boundary. At uniform density the strong bias of a smooth tangent with nonzero normal derivative has exact order $h^{7/4}$, with a coefficient universal across dimensions, given by the first unmatched Neumann jet of the input; the next term of the quadratic-form bias has order $h^{5/2}$. The proof uses a two-term boundary heat-kernel estimate with constant $\sqrt{2/\pi}$, from which the boundary spectral law follows by a Laplace Tauberian argument. On an interval, at every fixed smooth positive density, we compute the square-root boundary layer of the normalization, the exact $h^{7/4}$ profile from endpoint jets, and the weak h^2 bias, which equals $-\frac{1}{48}$ times the Wasserstein Hessian of Fisher information. Positive corrections uniformly form-comparable to the tensor attain the optimal strong order $(h/\log(1/h))^{7/4}$, with polylogarithmic rank on manifolds and rank at most eight at fixed density on the interval; corrections commuting with the Neumann Laplacian attain exactly $h^{7/4}$. On flat cylinders polylogarithmic rank is necessary among finite-rank perturbations of dominated spectral multipliers.

Keywords. Sinkhorn divergence, entropic optimal transport, heat kernel, Neumann boundary condition, boundary layer, Riemannian metric tensor, Fisher information.
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1 Introduction

The heat-kernel Sinkhorn tensor at time h converges, as $h \downarrow 0$, to the inverse of the weighted Neumann Laplacian, the Riemannian tensor of quadratic optimal transport. On boundaryless manifolds the leading bias has order h^2 and is governed by the Fisher information [11]. Lavenant, Luckhardt, Mordant, Schmitzer and Tamanini [7, Section 4.4] construct the Riemannian geometry of Sinkhorn divergences and compute formally, for the quadratic cost on a bounded convex domain, that the metric tensor tends to the transport tensor $\langle \dot\mu/\mu, L_\mu^{-1}[\dot\mu/\mu]\rangle$ as the regularization vanishes [7, (35)]. Comparing these two results identifies the question this article answers.

The question. For the intrinsic reflecting heat-kernel cost on a manifold with boundary, what is the exact leading bias of the Sinkhorn tensor, at which order in h, and which strong rate can positive corrections attain while remaining uniformly form-comparable to the tensor? This question arises from the comparison above: the boundaryless analysis [11] gives order h^2, and the formal computation of [7] identifies the limit on convex domains for the quadratic cost.

The answer. A reflecting boundary changes the boundaryless picture: a square-root layer forms at the boundary and the strong bias becomes $h^{7/4}$. For the intrinsic reflecting heat-kernel cost we prove the small-regularization limit, on the interval at every smooth positive density and on manifolds with boundary at uniform density, together with its exact leading boundary correction; this is the heat-kernel counterpart of the formal limit (35) of [7, Section 4.4]. A concrete consequence is Theorem 5: at every smooth positive density $f = q^2$ on the interval, the strong bias of a smooth tangent has exact order $h^{7/4}$ whenever, at some endpoint a, its inward derivative E_a or the product J_a of its value with the inward slope of $\log q$ is nonzero, so the boundary replaces the boundaryless rate h^2 by $h^{7/4}$. Theorem 7 shows that every positive correction uniformly dominated by the tensor has strong error of order at least $(h/\log(1/h))^{7/4}$ on each smooth tangent with nonzero normal derivative, and that this rate is attained.

Edition note. This second edition merges the Archive record *Sharp reflecting heat-kernel tensor bias on smooth manifolds with boundary*, edition 1 [13], into this article: it adds the jet law in every dimension (Theorem 1), a self-contained proof of the boundary heat estimate and of the boundary spectral law (Lemmas 1 and 2), the correction theory on manifolds (Theorem 7) and the cylinder

rank lower bound (Theorem 10), with Theorems 2 and 6 of the first edition, now Theorems 2 and 8, as the one-dimensional cases.

Main results.

- *Jet law* (Theorem 1). At uniform density on a manifold M with boundary, an input whose first $\ell - 1$ Neumann jets match has strong bias of exact order $h^{\ell+3/4}$ after the natural Taylor subtractions, with constant $\frac{\sqrt{2}}{4\pi}\|g\|^2_{L^2(\partial M)}$ times an explicit integral, where g is the first unmatched jet. The form constant is a value of $\Gamma(z)(1-2^{-z})\zeta(z)$.
- *Fixed-density interval theory* (Theorems 3, 4, 5 and 6). The normalization has a layer of exact size $\sqrt{h}$; the strong $h^{7/4}$ profile is given by the endpoint data E_a, J_a; the weak h^2 coefficient is $-\frac{1}{48}\operatorname{Hess}_{W_2} I(f)$, the boundary form of the periodic Fisher identity of [11].
- *Optimal corrections* (Theorems 7, 8 and 9). Under uniform form domination the optimal strong order is $(h/\log(1/h))^{7/4}$, attained with polylogarithmic rank, and corrections commuting with the spectral projections attain $h^{7/4}$.
- *Rank lower bound* (Theorem 10). On $[0,1]\times\mathbb{T}^{d-1}$ the rank order $(\log(1/h))^{7(d-1)/[4(r-3/2)]}$ is necessary for finite-rank perturbations of dominated operators commuting with the spectral projections.

Place in the literature. The boundary spectral input is the Neumann Kuznecov sum of Han, Hassell, Hezari and Zelditch [4, Theorem 6]; here it is derived directly from the heat estimate, which is the form the tensor requires. The heat estimate itself rests on Huynh's construction of the Neumann heat kernel on the heat space [5, Appendix B]. Marshall and Coifman [8] and Vaughn, Berry and Antil [15] study boundary effects in ambient diffusion-kernel constructions; we use the intrinsic reflecting heat kernel throughout, and the quantity studied is the singular-inverse transport tensor. The fixed-kernel normalization and density differentiation of Kessler, Magnet and Feydy [6] provide the general sensitivity context. The normalized graph Laplacian estimates of Cheng and Landa [1] concern the boundaryless manifold setting.

Related Archive records. The boundaryless parent is *Leading bias and positive correction of heat-kernel Sinkhorn metrics*, edition 5 [11]. Its Lemma 3.2 enters Section 8, and its form-comparable correction and Douglas range argument are the periodic counterparts of Part III. The same tensor is estimated from samples and computed in *Statistical inference for empirical heat-kernel Sinkhorn metrics*, edition 6 [14], *Fourier computation of empirical diffusion metrics*, edition 3 [10], and *Sharp minimax rates for inverse diffusion metric inference*, edition 1 [12].

Organization. Section 2 fixes the setting. Part I (Section 3) proves the jet law. Part II (Sections 4–8) is the fixed-density theory on the interval. Part III (Section 9) treats corrections, and Part IV (Section 10) proves the cylinder rank lower bound.

2 Setting

Let M be a connected, smooth compact Riemannian manifold of dimension d, with nonempty smooth boundary. Use its Riemannian volume and its induced boundary measure. Let $L = -\Delta_N/2$ be the Neumann Laplacian, nonnegative with orthonormal eigenfunctions ϕ_j and eigenvalues λ_j.

Let $H_h = e^{-hL}$. At the uniform probability density, $q = \mathrm{vol}(M)^{-1/2}$. Since the Neumann semigroup fixes constants, the symmetric Sinkhorn scale is one. On the orthogonal complement of the constant define

$$T_h = hH_h(I - H_h^2)^{-1}, \qquad G = \tfrac{1}{2}L^{-1}.$$

Extend both operators by zero on constants. This is the intrinsic reflecting heat-kernel tensor in the normalization of [11]. Write L_{int} for the differential expression $-\Delta/2$ without boundary conditions.

Set

$$\Phi(s) = \frac{1}{2\sinh s} - \frac{1}{2s} = \sum_{j\ge 1} c_j s^{2j-1} \quad \text{near zero}, \qquad \Phi_k(s) = \Phi(s) - \sum_{1\le j\le k} c_j s^{2j-1}.$$

The empty sum is zero, and $c_1 = -1/12$. For a smooth mean-zero η suppose $\partial_\nu L_{\mathrm{int}}^j \eta = 0$ on the boundary for $0 \le j < \ell - 1$, where $\ell \ge 1$. Set $g = \partial_\nu L_{\mathrm{int}}^{\ell-1}\eta$.

The interval model. In Part II we take $M = [0,1]$ and allow a nonuniform density. At $f = 1$ the tensors below are the tensors T_h, G above with $d = 1$.

Let $L = -\frac{1}{2}\partial_x^2$ with Neumann conditions on $[0,1]$, and $H_h = e^{-hL}$. Write $e_0 = 1$, $e_n = \sqrt{2}\cos(\pi n x)$ for $n \ge 1$, and $s_n = (\pi n)^2/2$. For a smooth strictly positive probability density f, set $q = \sqrt{f}$. Let $b_h > 0$ solve the symmetric normalization

$$b_h H_h(b_h q) = q, \qquad Q_h = M_{b_h} H_h M_{b_h}.$$

The operator Q_h fixes q. On $q^\perp$ define

$$T_h(f) = hQ_h(I - Q_h^2)^{-1}, \qquad G(f) = \tfrac{1}{2}S_f^\dagger, \quad \langle u, S_f v\rangle = \tfrac{1}{2}\int_0^1 f(u/q)'(v/q)'.$$

The form of S_f has domain $H^1(0,1)$; its kernel is the span of q. Both tensors are zero on that span. For smooth v in the operator domain the natural condition is $(v/q)' = 0$ at the two endpoints. This is the reflecting intrinsic heat cost and uses the local Sinkhorn tensor normalization of the periodic record [11], following the exact fixed-kernel geometry of [7]. Schur scaling is symmetric; the relevant Markov operator is unitarily conjugate to Q_h by multiplication by q.

For $r > 0$ put

$$t(r) = \frac{\tanh(r^2/4)}{r^2}, \quad \Psi(z) = \frac{2}{\pi}\int_0^\infty t(r)\cos(rz)\,dr, \quad F(r) = \frac{1}{2\sinh(r^2/2)} - \frac{1}{r^2}.$$

The function t is extended smoothly and evenly to the line, with $t(0) = 1/4$. At the endpoints use the outward derivative $\partial_\nu = -\partial_x$ at zero and $\partial_\nu = \partial_x$ at one.

3 Part I: the jet law at uniform density

The leading bias at uniform density is governed by the first unmatched normal derivative among the successive powers of the Neumann differential expression applied to the input. The theorem below gives the strong and quadratic-form asymptotics for every such order. Its spectral input is the boundary heat estimate of Lemma 1, and the main step is its transfer through the reflecting Sinkhorn multiplier, including the Taylor subtractions.

Theorem 1. *For $k = \lfloor \ell/2 \rfloor$,*

$$h^{-2\ell-3/2} \left\| (T_h - G)\eta - \sum_{j=1}^{k} c_j h^{2j} L^{2j-1}\eta \right\|_2^2 \longrightarrow \frac{\sqrt{2}}{4\pi} \|g\|^2_{L^2(\partial M)} \int_0^\infty \Phi_k(s)^2 s^{-2\ell-1/2}\, ds.$$

For the quadratic form,

$$h^{-2\ell-1/2} \left[\langle \eta, (T_h - G)\eta \rangle - \sum_{j=1}^{\ell} c_j h^{2j} \langle \eta, L^{2j-1}\eta \rangle \right] \longrightarrow \frac{\sqrt{2}}{4\pi} \|g\|^2_{L^2(\partial M)} \int_0^\infty \Phi_\ell(s) s^{-2\ell-1/2}\, ds.$$

The powers in the second formula are interpreted as spectral quadratic forms. Both integrals are finite and nonzero. For $\ell = 1$ these are the $h^{7/4}$ strong bias and $h^{5/2}$ form remainder.

For $d = 1$ and $\ell = 1$ the boundary is $\{0, 1\}$ and $\|g\|^2_{L^2(\partial M)} = E_\eta$. The substitution $s = r^2/2$ turns $\frac{2}{\pi} \int F(r)^2 r^{-4}\, dr$ into $\frac{\sqrt{2}}{4\pi} \int \Phi(s)^2 s^{-5/2}\, ds$, and likewise for the form integral, where $c_1 h^2 \langle \eta, L\eta \rangle = -\frac{h^2}{24} \|\eta'\|_2^2$. Theorem 1 therefore contains the following interval statement, which Section 3.4 proves directly at the lower regularity C^4.

Theorem 2. *If $f = 1$ and $\eta \in C^4([0, 1])$ has mean zero, write $E_\eta = |\eta'(0)|^2 + |\eta'(1)|^2$. Then*

$$h^{-7/2} \|(T_h(1) - G(1))\eta\|_2^2 \longrightarrow \frac{2E_\eta}{\pi} \int_0^\infty \frac{F(r)^2}{r^4}\, dr.$$

Also,

$$h^{-5/2} \left[\langle \eta, (T_h(1) - G(1))\eta \rangle + \frac{h^2}{24} \|\eta'\|_2^2 \right] \longrightarrow \frac{2E_\eta}{\pi} \int_0^\infty \frac{F(r) + r^2/24}{r^4}\, dr.$$

Both integrals are finite and strictly positive.

3.1 The boundary heat estimate

Write Γ for restriction to the boundary and p_t for the kernel of H_t with respect to the Riemannian volume. For $g \in L^2(\partial M)$ and $x \in \partial M$, $\Gamma H_t \Gamma^* g(x) = \int_{\partial M} p_t(x, y) g(y)\, d\sigma(y)$ for $t > 0$.

Lemma 1. *Uniformly for $0 < t \le 1$:*

1. *for $x, y \in \partial M$ and every N, $|p_t(x, y)| \le C_N t^{-d/2} (1 + \operatorname{dist}(x, y)/\sqrt{t})^{-N}$;*
2. *$\|\Gamma H_t \Gamma^*\|_{L^2(\partial M) \to L^2(\partial M)} \le C t^{-1/2}$;*
3. *for $g \in C^1(\partial M)$,*

$$\Gamma H_t \Gamma^* g = \sqrt{2/\pi}\, t^{-1/2} g + O_{L^\infty(\partial M)}(\|g\|_{C^1}).$$

Proof. We use the construction of the Neumann heat kernel on the heat space by Huynh [5, Appendix B], which follows Mazzeo and Vertman [9]. For functions the absolute Neumann condition is the Neumann condition. Huynh's normalization is $e^{s\Delta}$, and $H_t = e^{s\Delta}$ with $s = t/2$. By [5, Theorem 41] the kernel is polyhomogeneous on the heat space, with index set $\{j - d : j \in \mathbb{N}_0\}$ at the front face and empty index set at the temporal face. Concretely, in boundary normal coordinates $x = (x', x_n)$, $y = (y', y_n)$ near a boundary point, with $\tau = \sqrt{s}$, $\zeta' = (x' - y')/\tau$, $\xi_x = x_n/\tau$ and $\xi_y = y_n/\tau$, we have $s^{d/2} e^{s\Delta}(x, y) = \tilde{A}(\tau, x', \xi_x, \zeta', \xi_y)$, where $\tilde{A}$ is smooth up to $\tau = 0$ and, together with its τ-derivative, decays faster than any power of $|\zeta'|$, uniformly for bounded ξ_x, ξ_y. Away from the diagonal the kernel is $O(t^\infty)$ with all derivatives, uniformly on compact sets [5, (37)]. The leading front-face coefficient is the model in [5, Claim 42, (34)]: restricting the heat equation and the Neumann condition to the front face shows that this coefficient is the heat kernel of the frozen half-space Neumann model at $(x', 0)$, hence the sum of the direct and reflected Gaussians by uniqueness for that model. On the boundary $\xi_x = \xi_y = 0$ the direct and reflected Gaussians agree, and

$$\tilde{A}(0, x', 0, \zeta', 0) = 2(4\pi)^{-d/2} e^{-|\zeta'|^2_{\gamma(x')}/4},$$

where γ is the induced boundary metric.

Part 1 follows from the rapid decay in ζ' near the diagonal and the off-diagonal bound, by compactness of ∂M. Part 2 follows from Part 1 by the Schur test, since $\int_{\partial M} t^{-d/2}(1 + \operatorname{dist}(x, y)/\sqrt{t})^{-N}\, d\sigma(y) \le C t^{-1/2}$ for $N > d$. For Part 3 fix $x \in \partial M$ and a coordinate patch around it. The contribution of y outside the patch is $O(\|g\|_\infty)$ by Part 1. Inside the patch put $y' = x' - \tau\zeta'$ and let ω be the coordinate density of $d\sigma$. Up to a rapidly small term, the patch contribution is

$$\tau^{-1} \int_{\mathbb{R}^{d-1}} \tilde{A}(\tau, x', 0, \zeta', 0)\, g(x' - \tau\zeta')\, \omega(x' - \tau\zeta')\, d\zeta'.$$

Replace $\tilde{A}(\tau, \cdot)$ by $\tilde{A}(0, \cdot)$, $g(x' - \tau\zeta')$ by $g(x')$ and $\omega(x' - \tau\zeta')$ by $\omega(x')$. The three replacements cost $O(\tau\langle\zeta'\rangle^{-N})$, $O(\tau|\zeta'|\,\|g\|_{C^1})$ and $O(\tau|\zeta'|)$ times rapidly decaying factors, so after multiplication by τ^{-1} and integration they give $O(\|g\|_{C^1})$. The leading term is

$$\tau^{-1} g(x)\, 2(4\pi)^{-d/2} \int e^{-|\zeta'|^2_{\gamma(x')}/4} \omega(x')\, d\zeta' = \tau^{-1} g(x)\, 2(4\pi)^{-1/2} = \sqrt{2/\pi}\, t^{-1/2} g(x),$$

since $\omega(x')\, d\zeta'$ is the Riemannian measure of $\gamma(x')$ and $\tau = \sqrt{t/2}$. For $d = 1$ the integral over $\mathbb{R}^0$ is evaluation and gives the same constant. □

On $[0, 1]$ the estimate holds up to an exponentially small error: $p_t(0, 0) = 1 + 2\sum_{n \ge 1} e^{-t(\pi n)^2/2} = \sqrt{2/(\pi t)} + O(e^{-c/t})$ by theta inversion.

3.2 The boundary spectral measure

For $g \in L^2(\partial M)$ let

$$b_j(g) = \int_{\partial M} g\, \phi_j, \qquad \mu_g = \sum_{\lambda_j > 0} |b_j(g)|^2 \delta_{\lambda_j}.$$

For $t > 0$, $\langle g, \Gamma H_t \Gamma^* g\rangle = |b_0(g)|^2 + \int e^{-t\lambda}\, \mu_g(d\lambda)$.

Lemma 2. *For $g \in L^2(\partial M)$ and $R \ge 0$, $\mu_g([0, R]) \le C(1 + R)^{1/2}\|g\|_2^2$. For $g \in C^1(\partial M)$ the rescaled measures $\nu_h(A) = h^{1/2}\mu_g(h^{-1}A)$ converge vaguely on $[0, \infty)$ to $(\sqrt{2}/\pi)\|g\|_2^2 s^{-1/2}\, ds$, and if $g \ne 0$,*

$$\mu_g([0, R]) \sim \frac{2\sqrt{2}}{\pi} \|g\|_2^2 R^{1/2}, \qquad R \to \infty.$$

Proof. For $R \geq 1$, $\mu_g([0,R]) \leq e \int e^{-\lambda/R}\, d\mu_g \leq e\langle g, \Gamma H_{1/R}\Gamma^* g\rangle \leq CR^{1/2}\|g\|_2^2$ by Lemma 1(2); for $R < 1$ use the bound at $R = 1$. For the limit, Lemma 1(3) gives, for every $u > 0$,

$$\int e^{-us}\, \nu_h(ds) = h^{1/2}\big[\langle g, \Gamma H_{uh}\Gamma^* g\rangle - |b_0(g)|^2\big] \longrightarrow \sqrt{2/\pi}\, \|g\|_2^2 u^{-1/2} = \int_0^\infty e^{-us} \frac{\sqrt{2}}{\pi} \|g\|_2^2 s^{-1/2}\, ds.$$

By the continuity theorem for Laplace transforms of positive measures [3, Chapter XIII, Section 1], ν_h converges vaguely to the stated limit, which has no atoms. Evaluating on $[0,1]$ with $h = 1/R$ gives the cumulative asymptotic, which is the Hardy–Littlewood–Karamata theorem in this case. □

This normalization agrees with the Neumann Kuznecov sum $\sum_{\lambda_j' < \lambda} |\langle g, \phi_j\rangle|^2 = \frac{2}{\pi}\lambda\|g\|^2 + O_g(1)$ of [4, Theorem 6], stated in frequency λ_j' with $-\Delta\phi_j = \lambda_j'^2\phi_j$. Since $\lambda_j = \lambda_j'^2/2$, the count up to R has leading term $\frac{2}{\pi}\sqrt{2R}\,\|g\|^2 = \frac{2\sqrt{2}}{\pi}\|g\|^2 R^{1/2}$.

The cumulative upper bound supplies uniform integrability of ν_h for any test function with absolute envelope $O(s^\alpha)$ near zero for $\alpha > -1/2$ and $O(s^{-\beta})$ at infinity for $\beta > 1/2$, by dyadic decomposition at either end. Below the first positive eigenvalue the measures vanish, so the term $1 + R$ in the envelope is harmless after rescaling. Hence $\int \varphi\, d\nu_h \to (\sqrt{2}/\pi)\|g\|_2^2 \int \varphi(s) s^{-1/2}\, ds$ for every continuous φ on $(0,\infty)$ with these envelopes.

3.3 Transferring the law to the tensor

Green's formula gives, for smooth η,

$$\lambda_j\langle \phi_j, \eta\rangle = \langle \phi_j, L_{\mathrm{int}}\eta\rangle + \tfrac{1}{2} b_j(\partial_\nu \eta).$$

Applying this repeatedly under the stated compatibility conditions yields

$$\langle \phi_j, \eta\rangle = \lambda_j^{-\ell}\big[\langle \phi_j, L_{\mathrm{int}}^\ell \eta\rangle + \tfrac{1}{2} b_j(g)\big].$$

The interior coefficients here are square summable. In particular the domain conditions for each strong Taylor term hold: $2j - 1 \leq \ell - 1$ for $j \leq \lfloor \ell/2 \rfloor$. For each form term the spectral moment is finite by the cumulative boundary upper bound and $2j - 1 \leq 2\ell - 1$.

The remainder strong multiplier is $h\Phi_k(h\lambda_j)$. The boundary coefficient part of its squared norm is

$$\frac{1}{4} h^{2\ell+2} \int_0^\infty \Phi_k(s)^2 s^{-2\ell}\, \mu_g(d(s/h)).$$

The weights satisfy the required endpoint envelopes: the first unsuppressed Taylor order is at least ℓ, while the maximum power of the large-s polynomial is at most $\ell - 1$. The leading constant follows from rescaled boundary measure convergence. The interior term is $O(h^{\ell+1})$ in norm, since $s^{-\ell}\Phi_k(s)$ is bounded. The cross term is therefore negligible on the indicated squared-norm scale. In the form calculation the same expression uses $h^{2\ell+1}\Phi_\ell(s)s^{-2\ell}/4$ in place of the squared weight. The first unsuppressed Taylor order is $2\ell + 1$ and the last subtracted one is $2\ell - 1$, yielding the required envelopes at both ends. The weighted square of the interior coefficients is $O(h^{2\ell+1})$ using boundedness of $s^{-2\ell}\Phi_\ell(s)$. Use Cauchy–Schwarz with the absolute weight for the cross term. This proves the form limit.

The strong integral is strictly positive from its nonzero multiplier. For the weak integral put $z = 1/2 - 2\ell$. The Mellin transform of $1/(2\sinh s)$ for $\operatorname{Re} z > 1$ is $\Gamma(z)(1 - 2^{-z})\zeta(z)$ by its odd exponential series. Continue it by subtracting $1/(2s)$ and the Taylor terms through degree $2\ell - 1$ on $(0,1)$. At the indicated z, each of these subtracted powers is integrable against s^{z-1} on $(1,\infty)$. Its continued pole term equals minus that latter integral. Thus the fully subtracted integral in the theorem is exactly $\Gamma(z)(1 - 2^{-z})\zeta(z)$. It is nonzero, since at a negative noninteger half-integer the gamma function is finite and nonzero, and the zeta functional equation gives no zero there.

3.4 The interval case at C^4 regularity

At $f = 1$ the scale is one and $Q_h = H_h$. The multiplier of $T_h(1) - G(1)$ at frequency $n \geq 1$ is $hF(r_n)$, $r_n = \pi n\sqrt{h}$. The multiplier after adding the quadratic-form term $h^2 L/12$ is $h[F(r_n) + r_n^2/24]$. Let $\eta_n = \langle \eta, e_n \rangle$. The cosine-coefficient formula of Section 5, applied with $e_n = \sqrt{2}\cos(\pi n x)$, gives

$$\eta_n = \sqrt{2}[(-1)^n \eta'(1) - \eta'(0)](\pi n)^{-2} + O(n^{-4}).$$

The squared norm is the sum of $h^2 F(r_n)^2 |\eta_n|^2$. On each fixed positive range of r_n, replacing η_n by its leading term gives a relative-scale Riemann limit on each of the two parity sublattices, whose spacing is $2\pi\sqrt{h}$. A common envelope after dividing each summand by h^4 is $CF(r)^2/r^4$. This is bounded near zero and is $O(r^{-8})$ at infinity. The parity cross terms cancel, giving the first limit. For the form remainder, the corresponding summand after dividing by h^3 has envelope $C|F(r) + r^2/24|/r^4$. This is $O(r^2)$ at zero and $O(r^{-2})$ at infinity. The same parity Riemann sums give the second limit, since $\langle \eta, L\eta \rangle$ in the form sense equals $\|\eta'\|_2^2/2$. This form identity uses only the form domain of L. Positivity of the second integrand follows from $u/\sinh u > 1 - u^2/6$ for $u > 0$. For $u \geq \sqrt{6}$ it is immediate, and for smaller u it follows from comparison of the power series for $\sinh u/u$ and $(1 - u^2/6)^{-1}$. The first integrand is positive because $\sinh u > u$.

4 Part II: fixed densities on the interval

On $M = [0, 1]$ the density may be nonuniform. We first describe the square-root layer of the symmetric normalization, then the first density derivative of the strong bias at uniform density, then the exact strong profile at a fixed density, and finally the weak coefficient, which identifies the Fisher Hessian.

5 The normalization layer

Theorem 3. *Suppose q is strictly positive and C^4. There is a unique positive continuous normalization for small h, and uniformly on the interval,*

$$b_h = 1 + \frac{1}{q}(I - H_h)(I + H_h)^{-1} q + O(h).$$

For either endpoint a and $x_h = a - \nu(a)\sqrt{h}\, z$ with $z \geq 0$ in any fixed compact set,

$$\frac{b_h(x_h) - 1}{\sqrt{h}} \longrightarrow (\partial_\nu \log q)(a)\, \Psi(z).$$

In particular the uniform deviation is of exact order $\sqrt{h}$ when an endpoint slope is nonzero.

All convolution arguments in this section use even extensions to the circle of circumference two. The even extension of q is Lipschitz. Uniformly in h, $\|H_h q - q\|_\infty = O(\sqrt{h})$ and $\|[H_h, M_q]\|_{\infty\to\infty} = O(\sqrt{h})$, by the first absolute moment bound for the periodized Gaussian.

Write $R_h = (I + H_h)^{-1}$. It equals $I - M_h$, where $M_h = H_h(I + H_h)^{-1}$ has multiplier $m(\pi n\sqrt{h})$ on the doubled circle, with $m(r) = (1 + e^{r^2/2})^{-1}$. The inverse Fourier transform of m on the line is Schwartz. The convolution kernel of M_h is its scaled periodization, as follows by computing the Fourier coefficients of the periodized line kernel. The resulting uniform absolute kernel integral and first absolute moment give

$$\|R_h\|_{\infty\to\infty} \leq C, \qquad \|[R_h, M_v]\|_{\infty\to\infty} \leq C\sqrt{h}\ \mathrm{Lip}(v)$$

for a Lipschitz even-periodic multiplier v.

Write $b = 1 + w$ and divide $bH_h(bq) - q$ by q. The result is

$$E_h + (I + H_h)w + B_h(w), \qquad E_h = (H_h q - q)/q.$$

In fact the linear discrepancy is $wE_h + q^{-1}[H_h, M_q]w$, and the quadratic term is $wq^{-1}H_h(wq)$. Thus $\|B_h(w)\|_\infty \le C(\sqrt{h}\|w\|_\infty + \|w\|_\infty^2)$, with Lipschitz constant $O(\sqrt{h})$ on any fixed multiple of the $\sqrt{h}$-ball. The map $w \mapsto -R_h(E_h + B_h(w))$ contracts that ball into itself when its multiple is large enough and h is small. This constructs a positive scale and gives $w = -R_h E_h + O(h)$. Move q^{-1} outside R_h by the commutator estimate, which costs $O(h)$. This proves the uniform formula.

For uniqueness, if b_1, b_2 are two positive scales, their ratio u satisfies $uPu = 1$, where P is the Markov operator defined by the kernel H_h and $b_2 q$. If u is nonconstant, at every point Pu is strictly between its minimum and maximum. Evaluating at the two extrema of u would give opposite strict inequalities for the product of those extrema compared with one. If u is constant the equation gives $u = 1$.

For $v \in C^4([0,1])$ let $v_n = \int_0^1 v(x)\cos(\pi n x)\,dx$. Integration by parts gives

$$v_n = \frac{(-1)^n v'(1) - v'(0)}{(\pi n)^2} + O(n^{-4}).$$

Using $q = q_0 + 2\sum_{n\ge1} q_n \cos(\pi n x)$, the higher-order term contributes $O(h)$ uniformly to $(I - H_h)(I + H_h)^{-1}q$, since $\tanh(hs_n/2) \le \min(1, hs_n/2)$. At $x = \sqrt{h}\,z$, the term from $q'(0)$ is a Riemann sum with mesh $\pi\sqrt{h}$ for $-q'(0)\sqrt{h}\,\Psi(z)$. The term from $q'(1)$ has an extra $(-1)^n$ and its normalized sum tends to zero, by comparing the two parity Riemann sums. These limits hold uniformly for z in compact intervals by continuous dependence on z on bounded integration ranges and the common integrable tail of t. At $x = 1 - \sqrt{h}\,z$ the roles of the alternating and nonalternating terms are reversed. The uniform formula for b_h now gives the layer. Since t is positive, $\Psi(0) > 0$, proving the sharp uniform deviation.

6 Linearized tensor at uniform density

Theorem 4. *Let $a \in C^6([0,1])$ have mean zero and $f_\varepsilon = 1 + \varepsilon a$. For a finite mean-zero cosine polynomial η, set $D_h = \partial_\varepsilon(T_h(f_\varepsilon) - G(f_\varepsilon))|_0$, in the indicated fixed L^2 coordinates. Then*

$$h^{-7/2}\|D_h\eta\|_2^2 \longrightarrow \frac{|a'(0)\eta(0)|^2 + |a'(1)\eta(1)|^2}{8\pi} \int_0^\infty \Big(\coth(r^2/4)\,t''(r)\Big)^2\,dr.$$

The integral is finite and strictly positive. In particular even endpoint-compatible cosine inputs can violate an $O(h^2)$ strong error bound uniform through one density derivative.

For fixed h, differentiating the scale at $f_\varepsilon = 1 + \varepsilon a$ is justified by the implicit function theorem with derivative $I + H_h$. Since $q_0' = a/2$, the scale derivative is

$$u_h = \partial_\varepsilon b_h|_0 = \tfrac{1}{2}(I - H_h)(I + H_h)^{-1}a.$$

Here parameter derivatives carry the subscript zero, whereas ordinary derivatives of a are spatial. The derivative of Q_h is $M_{u_h}H_h + H_h M_{u_h}$.

For clarity the differentiation of the inverse metric is made in fixed L^2. The conjugated form of S_f has fixed domain H^1 and depends smoothly on f locally. The addition of the orthogonal projection onto q makes the form coercive by Poincaré, so it can be differentiated and inverted between

H^1 and its dual. The derivative of the projection has zero matrix entries between nonconstant base cosine modes. The differentiated form has entries

$$\langle e_n, S_0' e_m\rangle = -\tfrac{1}{4}\int_0^1 a'(e_n e_m)' = -\tfrac{1}{2}\int_0^1 a\, L(e_n e_m).$$

The boundary term vanishes because the derivative of the cosine product is zero at both ends. Thus for $n, m \geq 1$,

$$\langle e_n, G_0' e_m\rangle = \frac{\langle a, L(e_n e_m)\rangle}{4 s_n s_m}.$$

For Q_h, subtract its moving rank-one projection before applying the analytic function $z \mapsto hz/(1-z^2)$. The compact Markov operator has a positive gap below its simple fixed eigenvalue at any fixed $h > 0$; after projection subtraction the formula for the derivative is the divided difference of this analytic function. Smooth dependence for fixed h follows from that of the scale and the moving projection. At the nonconstant entries this gives

$$\langle e_n, T_h' e_m\rangle = hA(x,y)\langle u_h, e_n e_m\rangle, \quad A(x,y) = \frac{\cosh(x^2/2)+\cosh(y^2/2)}{2\sinh(x^2/2)\sinh(y^2/2)}.$$

Here $x = \pi n\sqrt{h}$ and $y = \pi m\sqrt{h}$. Indeed if $z_j = e^{-hs_j}$, the divided difference multiplied by $z_n + z_m$ is $h(z_n+z_m)(1+z_n z_m)/[(1-z_n^2)(1-z_m^2)]$, equal to $hA(x,y)$.

We estimate a fixed column m. Write $a_j = \int a\cos(\pi j x)\,dx$ and $J_j = (-1)^j a'(1) - a'(0)$. Six integrations by parts give

$$a_j = \frac{J_j}{(\pi j)^2} + \frac{(-1)^j a_3^+ - a_3^-}{(\pi j)^4} + O(j^{-6}), \qquad j \geq 1,$$

where $a_3^\pm$ are fixed signed third derivatives. The product $e_n e_m$ is the sum of the cosine modes $|n-m|$ and $n+m$, and the two indices have the same parity. For $n > 2m$, the contribution of the leading coefficients to $h^{-2}\langle e_n, D_h e_m\rangle$ is

$$\frac{J_{n+m}}{2}\,B(x,y), \qquad B(x,y) = A(x,y)[t(x+y)+t(x-y)] - \frac{2}{x^2y^2}.$$

For fixed $x > 0$, as $y \to 0$,

$$B(x,y) \longrightarrow \coth(x^2/4)\, t''(x).$$

To see the limit and obtain uniform bounds, put $X = x^2/2$, $Y = y^2/2$. The identity $t(x)(1+\cosh X)/\sinh X = x^{-2}$ gives the exact decomposition

$$\begin{aligned} B(x,y) = {} & A(x,y)[t(x+y)+t(x-y)-2t(x)] \\ & + x^{-2}[1/\sinh Y - 2/y^2] + t(x)\frac{\cosh Y - 1}{\sinh X \sinh Y}. \end{aligned}$$

For $x \geq 1$ and small y, the first term is $O(x^{-4})$ since $A = O(y^{-2})$ and $t'' = O(x^{-4})$. The other terms are $O(y^2 x^{-2})$. At small x, y, $x^2y^2B(x,y)$ has an analytic extension, is even separately, and vanishes on both axes by the same identity. It is therefore divisible by x^2y^2 and B is bounded. Together these facts give the common envelope $|B(x,y)| \leq C\min(1, x^{-2})$, and a locally uniform limit for $x > 0$.

We record why the sharp constant comes from the leading coefficients alone. For $z_\pm = x \pm y$ set $B_\pm = A(x,y)t(z_\pm) - 1/(x^2y^2)$. The fourth-order coefficient contribution, up to its bounded endpoint factors, is

$$h^2 \left\{ \frac{h}{x^2} B(x,y) + h \sum_{\pm} B_\pm (z_\pm^{-2} - x^{-2}) \right\}.$$

When $x \le 1$ and $x > 2y$, $|B_\pm| \le C(x/y + y^2/x^2)$. Indeed subtract $t(x)$, use $t'(r) = O(r^3)$ near zero, $A = O((x^2y^2)^{-1})$, and use the last two terms of the preceding decomposition for the remaining half correction. When $x \ge 1$ the analogous bound is $C(x^{-3}/y + y^2/x^2)$. As $|z_\pm^{-2} - x^{-2}| \le Cy/x^3$, the entire fourth-order contribution is bounded by Ch^2/n^2 in both ranges. For the $O(j^{-6})$ coefficients, the derivative multiplier difference before applying a_j is $O_m(h^2n^4)$ if $x \le 1$, by Taylor expansion of the hyperbolic functions, and $O_m(1)$ if $x \ge 1$, since $j = |n \pm m| \asymp n$. Thus their column norm is $O_m(h^2)$. The finitely many entries $n \le 2m$ are $O_m(h^2)$ by the same fixed-mode Taylor expansion. Finally, the constant output entry is $O_m(h^2)$: differentiate $(T_h(f_\varepsilon) - G(f_\varepsilon))q_\varepsilon = 0$, and use the $O(h^2)$ fixed-mode constant-density bias.

If η is a finite sum of base cosine modes, the leading output for each parity and each fixed $x > 0$ after dividing by h^2 is therefore

$$\frac{\coth(x^2/4)t''(x)}{2\sqrt{2}} \left[(-1)^n a'(1)\eta(1) - a'(0)\eta(0) \right].$$

The envelope established above is square integrable, and the lower-order columns have norm $O(h^2)$. Sum squares on the two parity sublattices of spacing $2\pi\sqrt{h}$ to obtain Theorem 4. The limiting multiplier tends to $-1/4$ at zero by expansion of t, so the integral is positive. At fixed input and output modes the derivative entries are instead $O(h^2)$. Fixed cosine contractions are $O(h^2)$; the $h^{7/4}$ strong term lives at output frequencies of order $h^{-1/2}$.

Naturally compatible moving tangents. Let η be a finite mean-zero cosine polynomial as above, and take the moving section

$$\eta_\varepsilon = q_\varepsilon \left[\eta - \int_0^1 f_\varepsilon \eta \right].$$

Then $\eta_\varepsilon \perp q_\varepsilon$ and $(\eta_\varepsilon/q_\varepsilon)' = 0$ at the endpoints for every ε. For the derivative at zero of $(T_h(f_\varepsilon) - G(f_\varepsilon))\eta_\varepsilon$, the same sharp limit as in Theorem 4 holds with profile

$$\coth(r^2/4)t''(r) + 2F(r)/r^2.$$

Indeed, the additional term in the derivative is $(T_h(1) - G(1))(a\eta/2 - \int a\eta)$. The constant is killed. The endpoint slopes of $a\eta/2$ are $a'\eta/2$, since the cosine tangent has zero endpoint derivatives. The constant-density Fourier multiplier therefore gives the stated extra profile, on the same two parity sublattices and with the same boundary factors. The new profile tends to $-1/3$ at zero, so its square integral is positive.

7 The exact profile at a fixed density

Write $\partial_{\text{in}} = -\partial_\nu$ for the inward derivative. The density and tangent are smooth on the closed interval, with arbitrary endpoint derivatives.

Theorem 5. *Let $f = q^2$ be smooth and strictly positive, and let η be smooth with $\eta \perp q$. For each endpoint a set $E_a = \partial_{\mathrm{in}}\eta(a)$ and $J_a = \eta(a)\partial_{\mathrm{in}} \log q(a)$. Then*

$$\lim_{h\downarrow 0} h^{-7/2}\|(T_h(f) - G(f))\eta\|_2^2 = \frac{1}{2\pi}\sum_{a=0,1}\int_0^\infty \left[2E_a\frac{F(r)}{r^2} + J_a \coth(r^2/4)t''(r)\right]^2 dr.$$

The limit is positive if and only if one of the pairs (E_a, J_a) is nonzero. In particular, every density with a nonzero endpoint slope has smooth tangents with strong bias of exact order $h^{7/4}$. The same conclusion holds for some tangents satisfying the natural endpoint condition $(\eta/q)' = 0$.

7.1 Localized normalization and residual

Put $\delta = \sqrt{h}$, let z_a be the inward distance from endpoint a divided by δ, and let K be time-one reflecting heat on $[0,\infty)$. Write $D_1 = Kz - z$ and $D_3 = Kz^3 - z^3 - 3z$. Both have Gaussian decay at infinity, by taking the difference between even reflection and the unreflected first and third Gaussian moments. The definitions of t, Ψ give

$$(I + K)\Psi = D_1, \qquad Kz^2 = z^2 + 1.$$

For the first identity, the even extension of D_1 equals the integral of whole-line heat kernels from time zero to one, by $(|z|)'' = 2\delta_0$. Its Fourier transform is $2(1-e^{-r^2/2})/r^2$, which equals $(1+e^{-r^2/2})2t(r)$. The even profile Ψ decays faster than any inverse power, by integrating its Fourier formula by parts.

Use $W_M(x) = \sum_a (1+z_a)^{-M}$. For arbitrarily large fixed M, the following elementary reflecting expansion holds for smooth v:

$$H_h v = v + \tfrac{h}{2}v'' + \tfrac{h^2}{8}v^{(4)} + \sum_a \left[\delta\,\partial_{\mathrm{in}}v(a)D_1(z_a) + \tfrac{\delta^3}{6}\partial_{\mathrm{in}}^3 v(a)D_3(z_a)\right] + O(\delta^5 W_M + \delta^6).$$

To verify it, extend v smoothly past the endpoints. Unreflected heat has the stated second-order expansion by Taylor and Gaussian moments. Compare the reflected input to the smooth extension outside each endpoint, in its inward local coordinate. Their difference is twice the odd Taylor terms through degree three plus $O(|x-a|^5)$. Convolving the linear and cubic odd terms on the outside half-line gives the two displayed defects. The remainder has the factor $\delta^5 W_M$ by Gaussian tails, with remote images smaller than any power of δ. All bounds are uniform for smooth bounded input families.

Every reflected Gaussian kernel and scaled Schwartz kernel used here is bounded on the sup spaces with weight $\delta^j W_M + \delta^k$, with constant independent of h for fixed M, j, k. Indeed the endpoint distances are Lipschitz, so $W_M(y) \le W_M(x)(1+|x-y|/\delta)^M$, and the kernels have uniformly bounded rescaled polynomial moments. The same argument on the doubled interval uses distance on the circle and the reflected endpoint sets. A Lipschitz multiplication commutator then gains a factor δ. We may decrease M by any prescribed fixed number when multiplying a boundary expression by powers of $1 + z_a$.

Put $V = q''/(2q)$, $B_0 = 1 - hV/2 + h^2V^2/8$, and $d_a = \partial_{\mathrm{in}}q(a)$, $c_a = q(a)$. There is a localized scale approximation

$$b_h = B_0 - \delta\sum_a (d_a/c_a)\Psi(z_a) + O(\delta^2 W_M + \delta^6).$$

Here and below a weighted O-bound is pointwise. For the bulk terms, the formal smooth heat expansion in $M_{B_0}H_h M_{B_0}$ is $I - hS + h^2S^2/2$, where $S = L + V$ denotes the interior differential expression. It fixes q to order $O(h^3)$ because $Sq = 0$. The leading reflecting discrepancy in the normalization equation is $\delta\sum_a d_a D_1(z_a)$. The two linear appearances of the proposed correction

cancel it by $(I+K)\Psi = D_1$. Replacing $q(x)$ by c_a in these localized terms costs $O(\delta^2 W_M)$, and the layer products and remaining heat defects have that same bound. Comparison of the half-line convolution of these profiles with reflected interval heat costs smaller than any polynomial order, since the profiles are rapidly decaying. Thus the displayed approximation solves the normalization equation up to $O(\delta^2 W_M + \delta^6)$. In the contraction from Section 5, R_h is uniformly bounded in the corresponding weighted sup norm, and the linear commutator and quadratic Lipschitz bounds are $O(\delta)$ there on the ordinary $O(\delta)$ ball. Subtract the approximate fixed-point equation from the exact one and absorb the error term to obtain the bound.

Write $\chi_h = (I+Q_h)^{-1}\eta$. We also have

$$\chi_h = \tfrac{1}{2}\eta + \tfrac{h}{4}S\eta - \tfrac{\delta}{2}\sum_a (E_a - J_a)\Psi(z_a) + O(\delta^2 W_M + \delta^4).$$

In fact the preceding scale approximation and the heat expansion give

$$Q_h\eta = \eta - hS\eta + \delta\sum_a (E_a - J_a)D_1(z_a) + O(\delta^2 W_M + \delta^4).$$

The bulk order is given by B_0; the leading layer is the reflecting defect of η, minus the two normalization factors $J_a(I+K)\Psi$. Apply $I+Q_h$ to the proposed approximation for χ_h. The smooth $hS\eta/4$ term has bulk value $hS\eta/2$ after that operation, with error $O(\delta^2 W_M + \delta^4)$. On the rapidly decaying profiles Q_h differs from K at the rescaled endpoint by $O(\delta W_M)$, using $b_h = 1 + O(\delta)$ and reflected kernel decay. The leading terms cancel, leaving precisely the stated error. Since $Q_h - H_h = O(\delta)$ on the weighted sup spaces, $(I+Q_h)^{-1}$ is bounded there by perturbation of R_h. This proves the approximation.

Set $\psi = S_f^\dagger \eta$, $w = \psi/q$, and

$$\mathfrak{r}_h = 2hQ_h\chi_h - (I - Q_h)\psi.$$

The weighted Poisson equation is $\eta = -qw''/2 - q'w'$, with $w' = 0$ at both endpoints. At an endpoint put $u = \partial_{\mathrm{in}}^2 w(a)$, $v = \partial_{\mathrm{in}}^3 w(a)$. Then $-c_a u/2 = \eta(a)$ and $c_a v + 3d_a u = -2E_a$. We claim

$$\mathfrak{r}_h = \delta^3 \sum_a R_a(z_a) + O(\delta^4 W_M + \delta^6), \qquad R_a = E_a(\Psi - D_3/3) + J_a(I+K)(z^2\Psi).$$

For the first term of $\mathfrak{r}_h$, use $2hQ_h\chi_h = 2h(\eta - \chi_h)$. The preceding approximation gives its bulk $h\eta - h^2 S\eta/2$, and layer $\delta^3(E_a - J_a)\Psi$. For the second term use $\ell = b_h q$ and

$$(I - Q_h)\psi = b_h[wH_h\ell - H_h(\ell w)].$$

To bound errors, replacing b_h, ℓ by their localized approximants costs $O(\delta^4 W_M + \delta^6)$. Indeed an inner error always multiplies $w(x) - w(y)$ under heat. For each localized endpoint factor this difference is bounded by $C\delta^2(1 + z_a(x) + z_a(y))^2$ because $w'(a) = 0$. Gaussian weighted bounds absorb the polynomial. A uniform $O(\delta^6)$ error needs no such gain. For the outer error, the same estimate bounds the local difference integral by $C\delta^2(1 + z_a(x))^2$, after Gaussian integration. This proves the error assertion.

The smooth part of the approximant gives the same bulk $h\eta - h^2 S\eta/2$ as the first term. Its reflecting defect at a, by the displayed heat expansion and $w'(a) = 0$, is

$$\delta^3\left[\frac{d_a u}{2} z^2 D_1 - \frac{c_a v + 3d_a u}{6} D_3\right](z_a).$$

Indeed the two linear defects have coefficients differing by $d_a(w(x) - w(a))$; their first nonzero difference is $d_a\delta^2 u z^2/2$. The cubic coefficients differ at the endpoint by $-c_a v - 3d_a u$. Replacing

ℓ in the inner term by its boundary layer adds $-\delta^3 d_a u[z^2 K\Psi - K(z^2\Psi)]/2$ at a. This follows by multiplying the layer $-\delta d_a \Psi$ by the quadratic difference of w. The outer layer acting on the leading bulk adds $-\delta^3 J_a \Psi$. All higher products and Taylor remainders are $O(\delta^4 W_M + \delta^6)$ by the same polynomial distance estimates. Use $D_1 = \Psi + K\Psi$ and the two Poisson identities to simplify the layer of $(I - Q_h)\psi$ to

$$\delta^3 \left[(E_a/3) D_3 - J_a((I + K)(z^2\Psi) + \Psi) \right](z_a).$$

Subtracting proves the residual formula.

7.2 A Neumann inverse bound

Let $P = q^{-1} Q_h q$, acting reversibly on $L^2(f\,dx)$, and let $H^2_N = \{u \in H^2 : u'(0) = u'(1) = 0\}$. For g of zero f-mean,

$$\|(I - P)^{-1} g\|_{L^2(f)} \le C \left(\|g\|_{L^2(f)} + h^{-1} \|fg\|_{(H^2_N)^*} \right).$$

Here the dual norm uses unweighted integration on the interval. To prove it, $Pu = H_h(\ell u)/(H_h \ell)$ with $\ell = b_h q$, and the Dirichlet form of $I - P$ is

$$\tfrac{1}{2} \iint H_h(x, y) \ell(x) \ell(y) |u(x) - u(y)|^2 \, dx \, dy.$$

Uniform positive upper and lower bounds on ℓ compare this form to that of $I - H_h$. The Neumann spectral gap and equivalence of weighted and unweighted centered variances give $\|(I - P)^{-1}\|_{L^2_0(f) \to L^2_0(f)} \le C/h$. Also $\ell = q^2/(H_h \ell)$ is uniformly Lipschitz. Indeed $\|\ell - q\|_\infty = O(\delta)$, so $\|\partial_x H_h(\ell - q)\|_\infty = O(1)$ by the Gaussian derivative bound, and $H_h q$ is uniformly Lipschitz on the doubled circle. The same argument or Gaussian cancellation gives $\|\partial_x^2 H_h \ell\|_\infty = O(\delta^{-1})$.

Put $y = P(I - P)^{-1} g$, so $y = P(y + g)$ and y is smooth with Neumann derivative. The equation gives

$$(I - H_h) y = (H_h \ell)^{-1} \{ H_h(\ell y) - (H_h \ell)(H_h y) + H_h(\ell g) \}.$$

The commutator in braces is the covariance double integral of the increments of ℓ, y with two copies of the heat kernel, divided by two. On the doubled circle its L^2 norm plus h times the L^2 norm of its second derivative is at most $Ch\|y'\|_2$. For each derivative of the kernels, use a Gaussian bound at twice the variance with factor $h^{-j/2}$. The two translation increments have product L^2 norm at most $C|u - v|^2 \|y'\|_2$ for shifts u, v; integration gives the factor h. The smoothed source $H_h(\ell g)$ has the same scaled norm bounded by $C\|g\|_2$, by its Fourier multiplier. Multiplication by $(H_h \ell)^{-1}$ preserves that scaled norm up to a constant, by its first and second derivative bounds and the Fourier inequality $\sqrt{h}\|u'\|_2 \lesssim \|u\|_2 + h\|u''\|_2$.

The Fourier inequality

$$h\|y''\|_2 \le C(\|(I - H_h) y\|_2 + h\|((I - H_h) y)''\|_2)$$

follows from $u \lesssim (1 + u)(1 - e^{-u})$ for $u \ge 0$. The preceding bounds give $h\|y''\|_2 \le C\|g\|_2 + Ch\|y'\|_2$. Absorb the first derivative by interpolation and use the gap bound for $h\|y\|_2$. This proves a uniform $L^2_0(f) \to H^2_N$ bound for $hP(I - P)^{-1}$. The operator is selfadjoint on the weighted space. Duality, subtracting the weighted mean of each dual test before applying the bound, gives the claimed inverse estimate from $(I - P)^{-1} = I + P(I - P)^{-1}$.

7.3 Recovering the boundary layer

Let U_a be the decaying solution of $(I-K)U_a = R_a$, defined by even Fourier transform. For $r \neq 0$, write $e = e^{-r^2/2}$. The even transform of $\Psi - D_3/3$ equals $-2/r^2 + 2t(r) + 4(1-e)/r^4$. This follows by transforming the even powers $|z|$ and $|z|^3$, whose second and fourth derivatives are $2\delta_0$ and $12\delta_0$, respectively, and by using that the localized profile is integrable. The transform is continuous at zero. For the second profile the transform is $-2(1+e)t''(r)$. Both numerators vanish to order at least two at zero, so after division by $1-e$ the even extension of U_a has Fourier transform

$$-4E_a F(r)/r^2 - 2J_a \coth(r^2/4)t''(r).$$

The singularities at zero cancel, and the multiplier and all its derivatives are integrable at infinity. Thus U_a decays rapidly by integration by parts. Put $v_h = \delta^3 \sum_a U_a(z_a)$. Reflecting heat and the scale bound give

$$(I - Q_h)v_h = \delta^3 \sum_a R_a(z_a) + O(\delta^4 W_M + \delta^6).$$

Indeed the difference from the half-line profile is smaller than any power due to the distant-endpoint tails; replacing either b_h by one costs $O(\delta^4 W_M)$. The projection of v_h onto q is $O(\delta^4)$ in L^2, by the integral of its decaying profiles.

Set $e_h = \mathfrak{r}_h - (I - Q_h)v_h$. In addition to the pointwise bound $O(\delta^4 W_M + \delta^6)$, this error has $\int_I q e_h = O(\delta^6)$ on either half interval I. We give the mass argument, since the inverse estimate uses the weighted mass of each half separately. For any smooth or bounded u, integration of

$$q(I - Q_h)(qu) = \ell[(H_h \ell)u - H_h(\ell u)]$$

over a set I is exactly its cross-cut integral

$$\int_{x \in I,\ y \notin I} H_h(x,y)\ell(x)\ell(y)[u(x) - u(y)]\, dx\, dy.$$

In particular the cross-cut integrals for v_h at the interior midpoint are smaller than every power of δ. Use $\mathfrak{r}_h = h\eta - h(I - Q_h)\chi_h - (I - Q_h)\psi$. On the left half interval $\int q\eta = -f(1/2)w'(1/2)/2$. Compare its two cross-cut integrals at $1/2$ with a smooth auxiliary periodic model whose q, w agree in a fixed neighborhood of that cut. Choose the auxiliary w to be constant near a second cut to its left and use the arc between those cuts. Then the auxiliary $\eta = S(qw)$ gives the same differential flux integral over the arc. The bulk scale approximations make the two b_h's differ by $O(\delta^6)$ in the matched neighborhood. The two χ_h's differ by $O(\delta^4)$ there, since their first two interior terms agree. The heat kernel at an interior cut is the same unreflected Gaussian up to superpolynomial error. Its cross-cut mass is $O(\delta)$, and the contribution outside the matched neighborhood is superpolynomial. Thus the flux for $\psi = qw$ matches to $O(\delta^6)$, and the flux for χ_h matches to $O(\delta^6)$ after its factor h. At the auxiliary second cut the w flux is superpolynomial, and the χ_h flux after multiplication by h is $O(\delta^7)$, since locally $\eta = 0$ and $\chi_h = O(\delta^4)$. The auxiliary periodic residual is $O(h^3)$: the pure smooth expansions of Q_h through h^2 and of χ_h through h give cancellation of the bulk h and h^2 terms above. Therefore the original half-mass is $O(\delta^6)$. Reflection proves the same bound on the other half.

For any $\varphi \in H^2_N$, subtract $\varphi(a)$ on the half interval adjacent to endpoint a. The bound $|\varphi(x) - \varphi(a)| \leq C \operatorname{dist}(x,a)^{3/2} \|\varphi''\|_2$ uses the zero endpoint derivative. The per-half masses and the pointwise error give $\|q e_h\|_{(H^2_N)^*} = O(\delta^6 + \delta^{13/2})$. Also $\|e_h\|_2 = O(\delta^{9/2} + \delta^6)$. Apply the inverse bound to e_h/q, which has zero f-mean. Since $2(I - Q_h)(T_h - G)\eta = \mathfrak{r}_h$, we conclude

$$\|2(T_h - G)\eta - v_h\|_2 = O(\delta^4) = o(\delta^{7/2}).$$

Here we included the $O(\delta^4)$ projection of v_h. The two endpoints are separated on the scaled coordinate, so their rapidly decaying profiles have negligible cross term after dividing the squared norm by δ^7. Even Fourier Plancherel gives the stated integral. The two multiplier shapes have different ratios at zero and infinity: the ratio of $\coth(r^2/4)t''(r)$ to $F(r)/r^2$ tends to 6 and -6, respectively. Thus their linear combination vanishes only when both coefficients vanish. At a nonzero slope, take a tangent nonzero near that endpoint and subtract an interior bump for orthogonality. For a naturally compatible tangent take q times a smooth function constant and nonzero near that endpoint and remove its weighted mean by an interior bump. Both choices give a nonzero J_a.

8 Weak bias and the Fisher Hessian

Against a fixed smooth tangent the bias has the smaller order h^2, with the following coefficient. Let $w_\eta = S_f^\dagger \eta/q$ and

$$\mathfrak{s}(\zeta,\eta) = \tfrac{1}{2}\int_0^1 f(\zeta/q)'(\eta/q)'.$$

This is the closed form of S_f on these smooth tangents.

Theorem 6. *For fixed smooth $\zeta, \eta \perp q$,*

$$\lim_{h\downarrow 0} h^{-2}\langle \zeta, (T_h(f) - G(f))\eta\rangle = \tfrac{1}{24}\int_0^1 fV''w'_\zeta w'_\eta - \tfrac{1}{12}\mathfrak{s}(\zeta,\eta) + \tfrac{1}{6}\sum_{a=0,1} \partial_{\mathrm{in}} \log q(a)\, \zeta(a)\eta(a).$$

For a diagonal contraction this is

$$-\tfrac{1}{48}\operatorname{Hess}_{W_2} I(f)[q\eta, q\eta], \qquad I(f) = \int_0^1 f|(\log f)'|^2,$$

where the Hessian is the second derivative along the one-dimensional smooth quadratic Wasserstein geodesic. Thus the periodic Fisher identity survives in weak form, with the natural interval geodesic and its boundary term.

For each input denote its residual from Section 7 by $\mathfrak{r}_{\eta,h}$, and its cumulative weighted residual by

$$\mathcal{F}_{\eta,h}(x) = \int_0^x q\mathfrak{r}_{\eta,h}.$$

It vanishes at both endpoints because the residual is orthogonal to q. For each fixed x in the open interval,

$$h^{-3}\mathcal{F}_{\eta,h}(x) \longrightarrow \frac{f(x)}{12}\left[(S\eta/q)'(x) - V''(x)w'_\eta(x)\right].$$

To see this, use the periodic cross-cut comparison from Section 7 at the arbitrary fixed interior point x in place of the midpoint. The flux discrepancy is $o(h^3)$, since the bulk scales near the cut agree to $O(\delta^6)$ and the χ_h's agree to $O(\delta^4)$, their respective cut integrals having mass $O(\delta)$, and the latter flux having prefactor h. At the auxiliary cut the w flux is superpolynomial and the $h\chi_h$ flux is $O(\delta^7)$. For the smooth periodic model, Lemma 3.2 of [11] gives, on fixed smooth inputs,

$$Q_h = I - hS + h^2S^2/2 + h^3(-S^3/6 + R) + o(h^3).$$

In density coordinates its operator R satisfies

$$q^{-1}R(qw) = -[fV''w']'/(12f).$$

Expanding $I + Q_h$ on smooth input also gives $\chi_h = \eta/2 + hS\eta/4 + O(h^3)$ in L^2; the quadratic coefficient cancels, and the norm of $(I + Q_h)^{-1}$ is bounded. Consequently the periodic residual divided by h^3 tends to $R(qw_\eta) - S^2\eta/6$. Multiplying this expression by q makes it a derivative with primitive $f[(S\eta/q)' - V''w'_\eta]/12$. At the auxiliary second cut this primitive is zero. This proves the pointwise limit with the stated sign.

Let $J^{\mathrm{int}}_{\eta,a}(z) = \int_0^z R_a(t)\,dt$ for the two layer profiles of Section 7. They are rapidly decaying because R_a is rapidly decaying and has zero mean, as its Fourier transform vanishes at zero. On the left half interval, subtract $\delta^4 q(0) J^{\mathrm{int}}_{\eta,0}(x/\delta)$ from $\mathcal{F}_{\eta,h}(x)$. The resulting error is bounded by

$$C\big(\delta^5(1 + x/\delta)^{-M} + \delta^6\big)$$

for each fixed M, with its own constant. Indeed start from the $O(\delta^6)$ signed mass at the midpoint and integrate the pointwise residual estimate backwards to x. After the leading profile, the localized integrand is $O(\delta^4 W_{M+1})$, also accounting for $q(x) - q(0)$ on the profile scale. Its remaining tail gives the displayed bound. The right half interval has the reflected envelope and the signed profile $-\delta^4 q(1) J^{\mathrm{int}}_{\eta,1}((1 - x)/\delta)$. These signs follow from the zero total mass.

Integrating by parts gives $\langle \psi_\zeta, \mathfrak{r}_{\eta,h}\rangle = -\int w'_\zeta \mathcal{F}_{\eta,h}$, where $\psi_\zeta = S_f^\dagger \zeta$. After removing the cumulative profiles, the normalized integrand converges to its bulk expression by dominated convergence up to a negligible localized error. To justify the latter statement, w'_ζ vanishes at the endpoints, so the $\delta^5 W_M$ envelope contributes $O(\delta^7)$ to the integral. The remaining bound is $O(\delta^6)$ uniformly, and the pointwise limit in the open interval applies; equivalently, first integrate on compact interior subintervals, then use this uniform bound and the localized estimate on the complementary collars. The cumulative profiles themselves contribute in the normalized limit

$$\sum_a \frac{q(a)\partial^2_{\mathrm{in}} w_\zeta(a)}{2} \int_0^\infty z^2 R_a(z)\,dz.$$

This follows by rescaling $w'_\zeta = \pm\delta z \partial^2_{\mathrm{in}} w_\zeta(a) + O(\delta^2 z^2)$ at the two ends and using $\int_0^\infty z J^{\mathrm{int}}_{\eta,a}(z)\,dz = -\frac{1}{2}\int_0^\infty z^2 R_a(z)\,dz$.

Finally, the exact inverse relation on $q^\perp$ gives

$$\begin{aligned} 2\langle \zeta, (T_h - G)\eta\rangle = h^{-1}\langle \psi_\zeta, \mathfrak{r}_{\eta,h}\rangle + \langle \chi_{\zeta,h}, \mathfrak{r}_{\eta,h}\rangle \\ + \tfrac{2}{h}\langle (T_h - G)\zeta, \mathfrak{r}_{\eta,h}\rangle. \end{aligned}$$

The last term is $O(\delta^5)$ by the strong estimate and the residual L^2 bound. The middle term is $O(\delta^5)$ as well: $\chi_{\zeta,h} = \zeta/2 + O(\delta)$ uniformly, the residual has $O(\delta^4)$ absolute integral, and its leading localized profile has zero integral at each endpoint. Its contraction with ζ therefore gains another factor of δ. We now evaluate half the previously computed limit $h^{-3}\langle \psi_\zeta, \mathfrak{r}_{\eta,h}\rangle$.

The expansion at frequency zero of $\widehat{R}_a$ from Section 7 is $r^2(E_a/12 + J_a/4) + O(r^4)$. Hence $\int_0^\infty z^2 R_a = -E_a/12 - J_a/4$. Since $q(a)\partial^2_{\mathrm{in}} w_\zeta(a) = -2\zeta(a)$, the boundary contribution after taking half is $\sum_a \zeta(a)(E_a/24 + J_a/8)$. Integrating the bulk term by parts, with $w'_\zeta = 0$ at the two endpoints, gives $\frac{1}{24}\int fV''w'_\zeta w'_\eta - \frac{1}{12}\int \zeta S\eta$. The latter integral is the interior differential expression. A further integration by parts gives

$$\int_0^1 \zeta S\eta = \mathfrak{s}(\zeta, \eta) + \tfrac{1}{2}\sum_a \zeta(a)(E_a - J_a).$$

The E_a's cancel from the weak coefficient and the J_a's add to $\frac{1}{6}\sum_a \zeta(a)J_a$. This proves the bilinear assertion.

To identify the Fisher Hessian, write $\ell = \log f$ and $v = w'_\eta/2$. Then $q\eta = -(fv)'$, $v = 0$ at both endpoints, and $x \mapsto x + tv(x)$ is the increasing Wasserstein displacement for small t. It fixes the two endpoints and leaves a smooth positive density on the interval. The change of variables formula gives

$$I(f_t) = \int_0^1 f\left[\frac{\ell'}{1+tv'} - \frac{tv''}{(1+tv')^2}\right]^2.$$

Differentiating twice and integrating the cross term yields

$$I''(0) = 2\int_0^1 f[(v'')^2 - 3\ell''(v')^2] + 6[f\ell'(v')^2]_0^1.$$

Also $\eta/q = -v' - \ell' v$ and $4V'' = \ell'''' + \ell'\ell''' + (\ell'')^2$. Expanding the square and integrating by parts using $v(0) = v(1) = 0$ gives

$$\int_0^1 f[(v'' + \ell' v' + \ell'' v)^2 - 4V''v^2] = \int_0^1 f[(v'')^2 - 3\ell''(v')^2] + [f\ell'(v')^2]_0^1.$$

At an endpoint $f(v')^2 = \eta^2$, so the boundary bracket is $-2\sum_a \eta(a)J_a$. Substitute these identities into the diagonal weak coefficient to obtain $-I''(0)/48$.

9 Part III: optimal positive corrections

The compact positive corrections of the periodic record [11] preserve the tensor form up to uniform two-sided comparison. Under the same requirement the next three theorems give the optimal strong order with a boundary: on manifolds at uniform density, on the interval with an explicit rank-four correction, and on the interval at a fixed density with rank eight. Douglas's range theorem [2] preserves the completed form space and the square-root range exactly as in the periodic record, so only the boundary-specific steps are written out.

Theorem 7. *If $0 \le C_h \le M_0 T_h$ uniformly in h, and a fixed smooth mean-zero η has $\partial_\nu \eta \not\equiv 0$, then*

$$\|(C_h - G)\eta\|_2 \gtrsim_{\eta, M_0} (h/\log(1/h))^{7/4}.$$

If C_h commutes with the spectral projections of L, the lower bound becomes $h^{7/4}$, and this order is attained: the diagonal operator A_h, equal to G at $0 < \lambda \le 1/h$ and to T_h above, satisfies $T_h \le A_h \le (\sinh 1)T_h$ and $\|A_h - G\|_{H^r \cap 1^\perp \to L^2} \lesssim h^{7/4}$ for $r > d/2 + 3$. There exist compact positive operators C_h with $cT_h \le C_h \le CT_h$ and

$$\|C_h - G\|_{H^r \cap 1^\perp \to L^2} \lesssim (h/\log(1/h))^{7/4}, \qquad r > d/2 + 3.$$

They are finite-rank perturbations of a diagonal operator. These Sobolev spaces impose no boundary condition.

The eigenvalues of T_h are $t_h(\lambda) = h/(2\sinh(h\lambda))$ off constants. For $C_h \le M_0 T_h$, form Cauchy–Schwarz gives

$$\|P_{>R} C_h \eta\|_2 \lesssim_\eta \sqrt{h}\, e^{-hR/2}$$

when $hR \ge 1$, where $P_{>R}$ is the projection onto eigenvalues of L above R. Here $T_h \le G$ bounds the other factor independently of h. Green's formula and Lemma 2 imply $\|P_{>R} G\eta\|_2 \asymp_\eta R^{-7/4}$ when

$\partial_\nu \eta \not\equiv 0$. Indeed its weighted boundary tail norm squared has order $R^{-7/2}$, whereas its interior tail norm is $O(R^{-2})$. Take R a large constant times $h^{-1}\log(1/h)$ and subtract the exponentially small C_h tail. If C_h commutes with the spectral projections, on every eigenspace above $R \asymp h^{-1}$ with a large fixed factor one has $C_h \le M_0 t_h(\lambda) I \le (4\lambda)^{-1} I$. Thus the tail error is at least half the G tail, and the stronger lower bound follows.

For the upper constructions let A_h be the diagonal operator of the theorem. On $0 < \lambda \le 1/h$ the ratio of the eigenvalues of G and T_h is $\sinh(h\lambda)/(h\lambda) \in [1, \sinh 1]$, which gives the two-sided comparison. Its error $(G - A_h)\eta$ is the tail $\mathbf{1}_{\lambda > 1/h}(L)(G - T_h)\eta$, and $0 \le G - T_h \le G$. By Green's formula the G coefficients of η are $(2\lambda_j^2)^{-1}[\langle \phi_j, L_{\mathrm{int}}\eta\rangle + \frac{1}{2} b_j(\partial_\nu \eta)]$, so with $R = 1/h$ the squared tail is at most a constant times $R^{-4}\|L_{\mathrm{int}}\eta\|_2^2 + \int_{(R,\infty)} \lambda^{-4}\, \mu_{\partial_\nu \eta}(d\lambda)$. Lemma 2 and dyadic summation bound the integral by $CR^{-7/2}\|\partial_\nu \eta\|_2^2$. The trace theorem gives $\|(A_h - G)\eta\|_2 \lesssim h^{7/4}\|\eta\|_{H^r}$, proving the attainment. Set $N = h^{-1}\log(1/h)$, let $\mathbf{1}_{\mathrm{mid}}$ select $1/h < \lambda \le N$, and define the boundary-to-volume operator

$$V_h = \tfrac{1}{2}\mathbf{1}_{\mathrm{mid}}(L)(G - T_h)L^{-1}\Gamma^*.$$

Its range is in a finite-dimensional spectral subspace. The boundary spectral upper bound and dyadic summation give

$$\|V_h\|_{L^2(\partial M) \to L^2(M)} \lesssim h^{7/4}, \qquad \|T_h^{-1/2} V_h\| \lesssim h^{3/4}.$$

In fact $G - T_h \le G$ on the selected band, so the sum of squared coefficients is bounded by the integral of $C\lambda^{-4}$ against μ_g there. For the second estimate, $T_h \ge h H_h$ and $e^{hN/2} = h^{-1/2}$.

Let $\tau = h^{1/8}$ and define the volume-to-boundary probe

$$W_\tau = \frac{\Gamma(H_\tau - H_{4\tau})}{\sqrt{2/\pi}\sqrt{\tau}}.$$

It annihilates constants. For $g = \partial_\nu \eta$, Green's formula gives

$$\partial_t \Gamma H_t \eta = -\Gamma H_t L_{\mathrm{int}}\eta - \tfrac{1}{2}\Gamma H_t \Gamma^* g.$$

Integrate between τ and 4τ. The boundary heat estimate and the Markov bound on $H_t L_{\mathrm{int}}\eta$ give

$$\|W_\tau \eta - g\|_{L^2(\partial M)} \lesssim \sqrt{\tau}\, \|\eta\|_{H^r}.$$

The trace estimate $\|\Gamma H_\tau\| \lesssim \tau^{-1/4}$ comes from the boundary heat norm at time 2τ by multiplying by its adjoint. Likewise $H_h^{-1/2}$ can be composed with the probe adjoint by leaving heat time at least $\tau/2$. It follows that

$$\|W_\tau\| \lesssim \tau^{-3/4}, \qquad \|W_\tau T_h^{-1/2}\| \lesssim h^{-1/2}\tau^{-3/4}.$$

All inverse-square-root expressions here are on $1^\perp$, with bounded extensions as shown.

Define $C_h = A_h + V_h W_\tau + W_\tau^* V_h^*$. The perturbation is selfadjoint and finite rank, and its relative form bound is $O(h^{1/4}\tau^{-3/4}) = o(1)$. Thus it gives positivity, compactness and two-sided form comparison. By Green's formula the intermediate-frequency error $G - A_h$ equals $V_h g$ up to $O(h^2\|L_{\mathrm{int}}\eta\|_2)$. The probe replaces g to the order just proved. The high tail $\lambda > N$ of this error is bounded by $CN^{-7/4}\|\eta\|_{H^r}$ by the boundary spectral upper bound and the square summable interior term. The low part already matches G.

We now bound the reciprocal correction. For a dyadic band $[R, 2R]$, $\|\Gamma \mathbf{1}_{[R,2R]}(L)\| \lesssim R^{1/4}$ by the cumulative boundary upper bound. The interior contribution to $V_h^*\eta$ on this band is bounded by

$CR^{-11/4}\|L_{\mathrm{int}}\eta\|_2$: Green's formula gives one λ^{-1}, and V_h^* contributes two more. The boundary contribution is bounded by $CR^{-5/2}\|g\|_2$, with two factors of the boundary restriction norm. Summing over $R \geq 1/h$ gives $\|V_h^*\eta\| \lesssim h^{5/2}\|\eta\|_{H^r}$. Hence the reciprocal term has error $O(\tau^{-3/4}h^{5/2}\|\eta\|_{H^r})$, smaller than the desired rate. The other errors are $O(h^2)$, $O(h^{7/4}\sqrt{\tau})$, and $O(N^{-7/4})$ times the input norm. This proves the matching construction.

9.1 Size of the correction space

The correction can be realized with a polylogarithmic rank. On the boundary let P_B^∂ be its intrinsic Laplace spectral projection through frequency B, with

$$B = (\log(1/h))^{7/[4(r-3/2)]}.$$

Replace V_hW_τ by $V_hP_B^\partial W_\tau$ and the other term by its adjoint. The trace theorem and the boundary spectral Sobolev norm give

$$\|(I - P_B^\partial)\partial_\nu\eta\|_2 \leq CB^{-(r-3/2)}\|\eta\|_{H^r}.$$

After applying V_h, this costs at most the target rate. Orthogonal projection does not increase the earlier form bounds or the reciprocal estimate. The resulting correction has rank at most $2\operatorname{rank} P_B^\partial = O(B^{d-1})$ by the ordinary boundary Weyl bound. For $d = 1$, the boundary space has finite dimension.

9.2 The interval with an explicit rank-four correction

For $d = 1$, Theorem 7 requires $r > d/2 + 3 = 7/2$, so it applies on H^4. The boundary space is two-dimensional, and the rank bound $2\operatorname{rank} P_B^\partial$ equals four. The following statement is that case, with the correction written in the Neumann basis, for ordinary H^4 tangents with arbitrary endpoint derivatives.

Theorem 8. *Work at $f = 1$ and on the mean-zero subspace. If $0 \leq C_h \leq M_0T_h(1)$ for a constant M_0 independent of h, then for any fixed smooth mean-zero η with a nonzero endpoint derivative,*

$$\|(C_h - G(1))\eta\|_2 \ \gtrsim_\eta \ (h/\log(1/h))^{7/4}.$$

If each C_h is diagonal in the Neumann basis, the lower bound strengthens to $h^{7/4}$. There exist compact positive corrections with

$$cT_h(1) \leq C_h \leq CT_h(1), \qquad \|C_h - G(1)\|_{H^4\cap 1^\perp \to L^2} \ \lesssim \ (h/\log(1/h))^{7/4}.$$

They can be formed from a diagonal comparison and a selfadjoint correction of rank at most four.

For the first lower bound put P_N for the orthogonal projection onto e_n, $n > N$. For any L^2 input, Cauchy–Schwarz for the positive form gives

$$\|P_NC_h\eta\|_2 \leq M_0\sqrt{\|P_NT_h(1)P_N\|}\sqrt{\langle\eta, T_h(1)\eta\rangle}.$$

Here the second factor is bounded, since $T_h(1) \leq G(1)$. Choose $N \asymp \sqrt{\log(1/h)/h}$ so that $hs_N \geq 4\log(1/h)$. The first factor is then $O(h^{5/2})$, by the exact heat-tensor eigenvalues. In contrast, the cosine coefficient asymptotic in Theorem 2 and $G(1)e_n = (\pi n)^{-2}e_n$ give $\|P_NG(1)\eta\|_2 \asymp_\eta N^{-7/2}$. At least one of the parity coefficients is nonzero. This yields the lower bound by the triangle

inequality. If C_h is diagonal, choose $N \asymp h^{-1/2}$ with a large fixed proportionality constant so that $M_0 h/(2\sinh(hs_n)) \le (\pi n)^{-2}/2$ for $n > N$. Then the coordinatewise error is at least half the corresponding $G(1)$ tail, giving the strengthened bound.

For the construction abbreviate the eigenvalues by $T_n = h/(2\sinh(hs_n))$ and $G_n = (\pi n)^{-2}$. Define a base diagonal operator A_h to have eigenvalue G_n when $n \le J$, where J is the largest integer with $hs_J \le 1$, and T_n otherwise. Then $T_h(1) \le A_h \le (\sinh 1) T_h(1)$. Take N to be the largest integer with $hs_N \le \log(1/h)$. For each parity p choose a positive k_p of that parity with $k_p \asymp h^{-1/8}$. For small h these satisfy $k_p \le J < N$. Set $w_p = (\pi k_p)^2 e_{k_p}/\sqrt{2}$ and define v_p by its coefficients

$$(v_p)_n = \frac{\sqrt{2}(G_n - T_n)}{(\pi n)^2} \mathbf{1}_{\{J<n\le N,\ n \text{ of parity } p\}}.$$

Use

$$C_h = A_h + \sum_p (v_p \otimes w_p + w_p \otimes v_p).$$

All corrections are selfadjoint, and A_h is a finite modification of the compact operator $T_h(1)$. The square-root inverse norms obey

$$\|T_h(1)^{-1/2} w_p\| = O(h^{-3/8}), \qquad \|T_h(1)^{-1/2} v_p\|^2 \le C h^{-1} e^{hs_N} \sum_{n>J} n^{-8} = O(h^{3/2}).$$

For the first bound $hs_{k_p} \to 0$, so $T_{k_p} \asymp k_p^{-2}$. For the second use $0 \le G_n - T_n \le G_n$. Thus the perturbation has form bound $O(h^{3/8}) T_h(1)$. This proves positivity, compactness, and uniform two-sided comparison for small h.

For $\eta \in H^4$ the cosine coefficient expansion used above has $O(n^{-4}\|\eta\|_{H^4})$ remainder by integration by parts, the endpoint trace bounds, and Cauchy–Schwarz. For each parity put $\ell_p(\eta) = (-1)^p \eta'(1) - \eta'(0)$. The low probe gives $\langle w_p, \eta\rangle = \ell_p(\eta) + O(k_p^{-2}\|\eta\|_{H^4})$. On $J < n \le N$ the correction therefore replaces the defect $(G_n - T_n)\langle e_n, \eta\rangle$ up to an error bounded in norm by $C(J^{-11/2} + k_p^{-2} J^{-7/2})\|\eta\|_{H^4}$, using $G_n - T_n \le G_n$. The reciprocal term $w_p \langle v_p, \eta\rangle$ at the low probe contributes at most $C k_p^2 \sum_{n>J} n^{-6} \|\eta\|_{H^4} = O(h^{9/4}\|\eta\|_{H^4})$. The uncorrected tail beyond N is $O(N^{-7/2}\|\eta\|_{H^4})$ and dominates these smaller errors. This proves the matching upper bound. As in the periodic setting, the form comparison preserves both the completed form space and the square-root range, the latter by Douglas's range theorem [2].

9.3 Form-comparable correction at fixed densities

The exact boundary profile also permits a sharp correction at every fixed smooth positive density, by a finite-rank perturbation of the tensor $T_h(f)$ itself.

Theorem 9. *Fix a smooth positive density f. On $q^\perp$ there exist compact positive C_h with $cT_h(f) \le C_h \le CT_h(f)$ and*

$$\|C_h - G(f)\|_{H^{10}\cap q^\perp \to L^2} \lesssim_f (h/\log(1/h))^{7/4}.$$

One can take $C_h - T_h(f)$ of rank at most eight. For any uniformly dominated positive correction $0 \le C_h \le M_0 T_h(f)$, and any fixed smooth tangent η for which $(S_f^\dagger \eta/q)'''$ is nonzero at an endpoint,

$$\|(C_h - G(f))\eta\|_2 \gtrsim_{f,\eta,M_0} (h/\log(1/h))^{7/4}.$$

In particular the operator rate is optimal under uniform form domination.

We retain f fixed and use the bounds of Section 7 uniformly in η, with constants times $\|\eta\|_{H^{10}}$. This follows from the one-dimensional Sobolev and trace bounds and the Poisson equation for w_η, which gives $w_\eta \in H^{12}$ with its norm bounded by $C_f\|\eta\|_{H^{10}}$ after fixing the weighted mean. The localized expansion uses at most six derivatives of η or qw_η and a bounded number of smoother coefficients depending only on f. In the auxiliary periodic comparison the smooth extensions near the fixed midpoint may be chosen by a fixed cutoff and multiplication, with these same bounds. All steps leading to the $O(h^2)$ strong remainder are linear in the input and satisfy these estimates.

There are four leading profile terms in Section 7, one for each endpoint and each of E_a, J_a. Write their sum for $(T_h - G)\eta$ as $\sum_{j=1}^4 V_j\mathcal{L}_j(\eta)$, with $\mathcal{L}_j$ the four indicated endpoint functionals and V_j the corresponding profiles of amplitude $\delta^3/2$. We may use the even, period-two periodizations of the profiles centered at their respective endpoints, with $O(\delta^M)$ change in L^2 for every fixed M. Then $\|V_j\|_2 = O(\delta^{7/2})$, $\|V_j\|_1 = O(\delta^4)$, and their cosine coefficients are bounded by $C\delta^4(1+\delta n)^{-4}$. The last statement follows from their explicit Fourier transforms in Section 7. Let N be the largest integer with $hs_N \le \log(1/h)$, and let $P_{\le N}$ be the Neumann projection through frequency N, including the constant. Write $\Pi^\perp$ for the projection onto $q^\perp$. Set $v_j = \Pi^\perp b_h P_{\le N} V_j$. Then

$$\|v_j - V_j\|_2 \lesssim (h/\log(1/h))^{7/4}.$$

Indeed the high cosine tail has this bound by summing the fourth-power coefficient decay. Multiplying by $b_h = 1 + O(\delta)$ adds only $O(\delta^{9/2})$. The removed q-component is $O(\delta^4)$: $\langle q, b_h P_{\le N} V_j\rangle = \langle P_{\le N}(qb_h), V_j\rangle$, and the cosine partial sums of $qb_h = \ell$ are uniformly bounded in sup norm. That bound follows by Cauchy–Schwarz from the uniform H^1 norm of ℓ in Section 7. The same reasoning yields

$$|\langle v_j, \eta\rangle| \le C\delta^4\|\eta\|_{H^{10}} \quad (\eta \perp q),$$

since $b_h\eta$ has uniformly bounded H^1 norm in terms of the input norm.

Use a probe frequency $\kappa \asymp h^{-1/16}$. For $\eta(a)$ the cosine projection kernel through frequency κ, evaluated at a, recovers the value to error $O(\kappa^{-1}\|\eta\|_{H^{10}})$; this follows by summing its n^{-2} coefficient bound. For the two inward derivatives, take the even and odd probes $(\pi k)^2 e_k/\sqrt{2}$ with $k \asymp \kappa$ of the two parities, and take their fixed linear combinations to isolate the two endpoints. The boundary expansion for cosine coefficients gives error $O(\kappa^{-2}\|\eta\|_{H^{10}})$. Include the factor $\partial_{\mathrm{in}} \log q(a)$ in the value probes for J_a. Let p_j be the resulting probes and $w_j = \Pi^\perp b_h p_j$. Their norms are $O(\kappa^2)$, all frequencies are $O(\kappa)$, and

$$|\langle w_j, \eta\rangle - \mathcal{L}_j(\eta)| \le C(\kappa^{-1} + \delta\kappa^2)\|\eta\|_{H^{10}},$$

using $b_h = 1 + O(\delta)$ and $\eta \perp q$.

Define $C_h = T_h(f) - \sum_j (v_j \otimes w_j + w_j \otimes v_j)$. For any trigonometric polynomial p, on the tangent space,

$$|\langle \Pi^\perp b_h p, \xi\rangle| \le \|H_h^{-1/2} p\|_2 \, h^{-1/2} \langle \xi, T_h(f)\xi\rangle^{1/2}.$$

This follows from $Q_h = b_h H_h b_h$ and $T_h(f) \ge hQ_h$ as forms on $q^\perp$. For v_j the factor on the right is $O(h^{-1/2}e^{hs_N/2}h^{7/4}) = O(h^{3/4})$. For w_j it is $O(h^{-1/2}\kappa^2) = O(h^{-5/8})$. Their products give a relative form perturbation $O(h^{1/8})$. This gives compactness, positivity, and uniform two-sided comparison for all small h. The outputs $v_j\langle w_j, \eta\rangle$ cancel the leading profile up to the Fourier tail, the smaller probe errors, and the $O(h^2)$ strong remainder. The reciprocal outputs $w_j\langle v_j, \eta\rangle$ cost $O(\kappa^2\delta^4\|\eta\|_{H^{10}}) = O(h^{15/8}\|\eta\|_{H^{10}})$. All these errors apart from the Fourier tail are $o((h/\log(1/h))^{7/4})$. This proves the upper bound.

For the lower bound the order-h gap of $I - Q_h$ on $q^\perp$ from Section 7 implies $T_h(f) \le CQ_h$ as forms, extending $T_h(f)$ by zero on q. Thus for any high-mode $g = P_{>N}g$, $\langle b_h^{-1}g, C_h b_h^{-1}g\rangle \le$

$C\langle g, H_h g\rangle$. Form Cauchy–Schwarz and the uniform bound on $\|T_h(f)\|$ give

$$\|P_{>N} b_h^{-1} C_h \eta\|_2 \le C_\eta \exp(-h s_N/2).$$

Here take $N \asymp \sqrt{\log(1/h)/h}$ with a sufficiently large fixed proportionality factor.

In contrast $b_h^{-1} G(f)\eta = (H_h \ell) w_\eta/2$. Use exponential Fourier coefficients of the even extensions to the doubled circle. For $w = w_\eta$ we have $w'(0) = w'(1) = 0$ and

$$w_n = \frac{w'''(0) - (-1)^n w'''(1)}{(\pi n)^4} + O(n^{-6}), \qquad n > 0.$$

Put $A = H_h \ell$. Its coefficients are $A_m = e^{-h s_{|m|}} \ell_m$. The uniform H^1 bound for ℓ gives $\sum_m |\ell_m| \le C$ and $\sum_m |m A_m| \le C\delta^{-1/2}$, the latter by Cauchy–Schwarz with the Gaussian multipliers. In the Fourier convolution of A and w, for $|m| \le n/2$ replace $(n-m)^{-4}$ by n^{-4}; the summed error is $O(n^{-5}\delta^{-1/2})$. For the complementary indices the sum is $O(e^{-h s_{\lfloor n/2 \rfloor}})$ by boundedness of the w coefficients and summability of ℓ_m. For sufficiently large proportionality factor in N this is $o(n^{-4})$ uniformly on $n > N$. Summing the constant and parity coefficients of A therefore gives

$$(Aw)_n = \frac{A(0) w'''(0) - (-1)^n A(1) w'''(1)}{(\pi n)^4} + o(n^{-4}) \quad \text{uniformly on } n > N.$$

Since $A(a) = q(a)/b_h(a) \to q(a) > 0$, at least one parity has a nonzero limiting coefficient when the stated third derivative condition holds. The norm of the $G(f)$ tail after multiplication by b_h^{-1} is of order $N^{-7/2}$, dominating the C_h tail for the chosen N. Uniform invertibility of the multiplication yields the claimed lower bound. There exist tangents satisfying the third derivative condition, since one can choose smooth w with zero first but nonzero third endpoint derivative and take $\eta = S_f(qw)$, which is automatically orthogonal to q.

10 Part IV: a rank lower bound on flat cylinders

Theorem 10. *On the flat cylinder $M = [0,1] \times \mathbb{T}^{d-1}$, $d \ge 2$, the rank order B^{d-1} of Section 9.1 is necessary among finite-rank perturbations of uniformly dominated operators that commute with the spectral projections of L. More precisely, suppose A_h commutes with the spectral projections of L, $0 \le A_h \le M_0 T_h$ uniformly, and C_h has the strong upper rate in Theorem 7 for a given $r > d/2 + 3$. Then*

$$\operatorname{rank}(C_h - A_h) \gtrsim (\log(1/h))^{7(d-1)/[4(r-3/2)]}.$$

The implicit constant can depend on the constant in the strong upper estimate and on M_0.

To prove it, choose a smooth function ξ supported near the left edge of the half-line with $\xi'(0) \ne 0$, smooth up to that edge. For a large parameter B_*, consider the tangents

$$B_*^{1/2-r} \xi(B_* x) e^{2\pi i k \cdot y}, \qquad B_* \le |k| \le 2B_*,$$

or their real span. The tangential frequencies are nonzero, so the tangents have mean zero. Their span has dimension $\asymp B_*^{d-1}$. The H^r norm of a linear combination is bounded by a constant times the coefficient ℓ^2 norm. For instance extend ξ smoothly and compactly to the line and use the Fourier Sobolev norm on the product of the line and the torus, then restrict. The normal scaling $B_*^{1/2-r}$ compensates for the derivative weights B_*^r and the support width.

Uniformly for normal cosine frequency $n \gg B_*$, the coefficient of each tangent on its tangential frequency is a nonzero constant times $B_*^{3/2-r} n^{-2}$, with relative error $O((B_*/n)^2)$. This is obtained

by two integrations by parts for the endpoint derivative and two more for the third derivative and the integral remainder. Take n in a band of fixed proportional width above a sufficiently large multiple of $h^{-1/2}$. On this band the fixed upper bound for A_h in the spectral eigenspaces implies a norm error at least half the G contribution, as in the commuting lower bound. Equivalently, take an eigenvalue tail containing this band, so any mixing within an eigenspace does not affect the estimate. The modes from distinct tangential frequencies are orthogonal in the G tail, and the eigenvalues of G there have order n^{-2} when B_* is polylogarithmic. Thus every such linear combination has $(A_h - G)$ tail norm bounded below by a constant times $h^{7/4}B_*^{3/2-r}$ times its coefficient norm.

If the corrective rank is below the dimension of this span, some nonzero vector in it belongs to the kernel of $C_h - A_h$, so the same lower bound applies to $C_h - G$. Taking B_* a sufficiently small fixed multiple of $(\log(1/h))^{7/[4(r-3/2)]}$ contradicts the assumed upper constant. This proves the rank lower bound.

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