Boundary layers in reflected heat-kernel Sinkhorn metrics

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Luminary Oddities

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Abstract

The reflecting heat kernel on an interval produces a square-root boundary layer in the symmetric Sinkhorn normalization at a nonzero density slope. At every fixed smooth positive density, we compute the exact strong $h^{7/4}$ tensor bias from the endpoint jets of the density and tangent. Fixed smooth contractions instead have weak h^2 bias given by the Wasserstein Hessian of Fisher information, with an explicit endpoint term in its differential-form representation. We also obtain the exact first density derivative of the strong boundary profile and the next quadratic-form remainder at uniform density. Compact positive finite-rank corrections attain the optimal form-comparable rate $(h/\log(1/h))^{7/4}$. At uniform density no diagonal correction gains the logarithmic factor.

1 Interval model and statements

At a fixed smooth positive density, the leading strong bias is a boundary effect. Theorem 4 gives its exact coefficient from the endpoint derivatives of the density and the tangent. Theorem 5 identifies the weaker order-h^2 pairing with the Wasserstein Hessian of Fisher information. The endpoint term in that theorem is part of the differential-form representation of the Hessian. The normalization layer and the uniform-density calculation below prepare the fixed-density proof.

Theorems 6 and 7 quantify the gain possible under uniform domination in the original tensor's quadratic form. The corrected operators remain positive and compact. At fixed density, the difference from the original tensor has rank at most eight.

Let $L = -\frac{1}{2}\partial_x^2$ with Neumann conditions on $[0,1]$, and $H_h = e^{-hL}$. Write $e_0 = 1$, $e_n = \sqrt{2}\cos(\pi n x)$ for $n \geq 1$, and $s_n = (\pi n)^2/2$. For a smooth strictly positive probability density f, set $q = \sqrt{f}$. Let $b_h > 0$ solve the symmetric normalization

$$b_h H_h(b_h q) = q, \qquad Q_h = M_{b_h} H_h M_{b_h}.$$

The operator Q_h fixes q. On $q^\perp$ define

$$T_h(f) = hQ_h(I - Q_h^2)^{-1}, \qquad G(f) = \tfrac{1}{2}S_f^\dagger, \quad \langle u, S_f v\rangle = \tfrac{1}{2}\int_0^1 f(u/q)'(v/q)'.$$

The form of S_f has domain $H^1(0,1)$; its kernel is the span of q. Both tensors are zero on that span. For smooth v in the operator domain the natural condition is $(v/q)' = 0$ at the two endpoints. This is the reflecting intrinsic heat cost and uses the local Sinkhorn tensor normalization of the periodic companion [1], following the exact fixed-kernel geometry of [2]. Schur scaling is symmetric; the relevant Markov operator is unitarily conjugate to Q_h by multiplication by q.

For $r > 0$ put

$$t(r) = \frac{\tanh(r^2/4)}{r^2}, \quad \Psi(z) = \frac{2}{\pi}\int_0^\infty t(r)\cos(rz)\,dr, \quad F(r) = \frac{1}{2\sinh(r^2/2)} - \frac{1}{r^2}.$$

The function t is extended smoothly and evenly to the line, with $t(0) = 1/4$. At the endpoints use the outward derivative $\partial_\nu = -\partial_x$ at zero and $\partial_\nu = \partial_x$ at one.

Theorem 1. *Suppose q is strictly positive and C^4. There is a unique positive continuous normalization for small h, and uniformly on the interval,*

$$b_h = 1 + \frac{1}{q}(I - H_h)(I + H_h)^{-1}q + O(h).$$

For either endpoint a and $x_h = a - \nu(a)\sqrt{h}\,z$ with $z \geq 0$ in any fixed compact set,

$$\frac{b_h(x_h) - 1}{\sqrt{h}} \longrightarrow (\partial_\nu \log q)(a)\,\Psi(z).$$

In particular the uniform deviation is of exact order $\sqrt{h}$ when an endpoint slope is nonzero.

Theorem 2. *If $f = 1$ and $\eta \in C^4([0,1])$ has mean zero, write $E_\eta = |\eta'(0)|^2 + |\eta'(1)|^2$. Then*

$$h^{-7/2}\|(T_h(1) - G(1))\eta\|_2^2 \longrightarrow \frac{2E_\eta}{\pi}\int_0^\infty \frac{F(r)^2}{r^4}\,dr.$$

Also,

$$h^{-5/2}\left[\langle \eta, (T_h(1) - G(1))\eta\rangle + \frac{h^2}{24}\|\eta'\|_2^2\right] \longrightarrow \frac{2E_\eta}{\pi}\int_0^\infty \frac{F(r) + r^2/24}{r^4}\,dr.$$

Both integrals are finite and strictly positive.

Theorem 3. *Let $a \in C^6([0,1])$ have mean zero and $f_\varepsilon = 1 + \varepsilon a$. For a finite mean-zero cosine polynomial η, set $D_h = \partial_\varepsilon(T_h(f_\varepsilon) - G(f_\varepsilon))|_0$, in the indicated fixed L^2 coordinates. Then*

$$h^{-7/2}\|D_h\eta\|_2^2 \longrightarrow \frac{|a'(0)\eta(0)|^2 + |a'(1)\eta(1)|^2}{8\pi}\int_0^\infty \left(\coth(r^2/4)\,t''(r)\right)^2\,dr.$$

The integral is finite and strictly positive. In particular even endpoint-compatible cosine inputs can violate an $O(h^2)$ strong error bound uniform through one density derivative.

The companion's periodic estimates assume smooth extensions across every coordinate seam. On an interval the reflecting kernel is the restriction of the heat kernel on the doubled circle to even functions. A smooth density on the interval need not have a smooth even extension. Marshall and Coifman [3] and Vaughn, Berry and Antil [4] study boundary effects in ambient diffusion-kernel constructions. We use the intrinsic reflecting heat kernel, and the quantity in Theorems 2 and 3 is the singular-inverse transport tensor. The fixed-kernel normalization and density differentiation of Kessler, Magnet and Feydy [5] provide the general sensitivity context. The normalized graph Laplacian estimates of Cheng and Landa [7] are for the boundaryless manifold setting. The results here do not require an ambient truncation of a Gaussian kernel at the endpoints.

2 The normalization layer

All convolution arguments in this section use even extensions to the circle of circumference two. The even extension of q is Lipschitz. Uniformly in h, $\|H_h q - q\|_\infty = O(\sqrt{h})$ and $\|[H_h, M_q]\|_{\infty\to\infty} = O(\sqrt{h})$, by the first absolute moment bound for the periodized Gaussian.

Write $R_h = (I + H_h)^{-1}$. It equals $I - M_h$, where $M_h = H_h(I + H_h)^{-1}$ has multiplier $m(\pi n\sqrt{h})$ on the doubled circle, with $m(r) = (1 + e^{r^2/2})^{-1}$. The inverse Fourier transform of m on the line is Schwartz. The convolution kernel of M_h is its scaled periodization, as follows by computing the Fourier coefficients of the periodized line kernel. The resulting uniform absolute kernel integral and first absolute moment give

$$\|R_h\|_{\infty\to\infty} \le C, \qquad \|[R_h, M_v]\|_{\infty\to\infty} \le C\sqrt{h}\ \mathrm{Lip}(v)$$

for a Lipschitz even-periodic multiplier v.

Write $b = 1 + w$ and divide $bH_h(bq) - q$ by q. The result is

$$E_h + (I + H_h)w + B_h(w), \qquad E_h = (H_h q - q)/q.$$

In fact the linear discrepancy is $wE_h + q^{-1}[H_h, M_q]w$, and the quadratic term is $wq^{-1}H_h(wq)$. Thus $\|B_h(w)\|_\infty \le C(\sqrt{h}\|w\|_\infty + \|w\|_\infty^2)$, with Lipschitz constant $O(\sqrt{h})$ on any fixed multiple of the $\sqrt{h}$-ball. The map $w \mapsto -R_h(E_h + B_h(w))$ contracts that ball into itself when its multiple is large enough and h is small. This constructs a positive scale and gives $w = -R_h E_h + O(h)$. Move q^{-1} outside R_h by the commutator estimate, which costs $O(h)$. This proves the uniform formula.

For uniqueness, if b_1, b_2 are two positive scales, their ratio u satisfies $uPu = 1$, where P is the Markov operator defined by the kernel H_h and $b_2 q$. If u is nonconstant, at every point Pu is strictly between its minimum and maximum. Evaluating at the two extrema of u would give opposite strict inequalities for the product of those extrema compared with one. If u is constant the equation gives $u = 1$.

For $v \in C^4([0,1])$ let $v_n = \int_0^1 v(x)\cos(\pi n x)\,dx$. Integration by parts gives

$$v_n = \frac{(-1)^n v'(1) - v'(0)}{(\pi n)^2} + O(n^{-4}).$$

Using $q = q_0 + 2\sum_{n\ge1} q_n \cos(\pi n x)$, the higher-order term contributes $O(h)$ uniformly to $(I - H_h)(I + H_h)^{-1}q$, since $\tanh(hs_n/2) \le \min(1, hs_n/2)$. At $x = \sqrt{h}\,z$, the term from $q'(0)$ is a Riemann sum with mesh $\pi\sqrt{h}$ for $-q'(0)\sqrt{h}\,\Psi(z)$. The term from $q'(1)$ has an extra $(-1)^n$ and its normalized sum tends to zero, by comparing the two parity Riemann sums. These limits hold uniformly for z in compact intervals by continuous dependence on z on bounded integration ranges and the common integrable tail of t. At $x = 1 - \sqrt{h}\,z$ the roles of the alternating and nonalternating terms are reversed. The uniform formula for b_h now gives the layer. Since t is positive, $\Psi(0) > 0$, proving the sharp uniform deviation.

3 Constant density and boundary tangents

At $f = 1$ the scale is one and $Q_h = H_h$. The multiplier of $T_h(1) - G(1)$ at frequency $n \ge 1$ is $hF(r_n)$, $r_n = \pi n\sqrt{h}$. The multiplier after adding the quadratic-form term $h^2 L/12$ is $h[F(r_n) + r_n^2/24]$. Let $\eta_n = \langle \eta, e_n\rangle$. The preceding integration by parts formula gives

$$\eta_n = \sqrt{2}[(-1)^n\eta'(1) - \eta'(0)](\pi n)^{-2} + O(n^{-4}).$$

The squared norm is the sum of $h^2F(r_n)^2|\eta_n|^2$. On each fixed positive range of r_n, replacing η_n by its leading term gives a relative-scale Riemann limit on each of the two parity sublattices, whose spacing is $2\pi\sqrt{h}$. A common envelope after dividing each summand by h^4 is $CF(r)^2/r^4$. This is bounded near zero and is $O(r^{-8})$ at infinity. The parity cross terms cancel, giving the first limit. For the form remainder, the corresponding summand after dividing by h^3 has envelope $C|F(r)+r^2/24|/r^4$. This is $O(r^2)$ at zero and $O(r^{-2})$ at infinity. The same parity Riemann sums give the second limit, since $\langle\eta, L\eta\rangle$ in the form sense equals $\|\eta'\|_2^2/2$. The stronger operator-domain condition on η is not needed for this form identity. Positivity of the second integrand follows from $u/\sinh u > 1-u^2/6$ for $u>0$. For $u\ge\sqrt{6}$ it is immediate, and for smaller u it follows from comparison of the power series for $\sinh u/u$ and $(1-u^2/6)^{-1}$. The first integrand is positive because $\sinh u > u$.

4 Linearized tensor at uniform density

For fixed h, differentiating the scale at $f_\varepsilon = 1+\varepsilon a$ is justified by the implicit function theorem with derivative $I+H_h$. Since $q_0' = a/2$, the scale derivative is

$$u_h = \partial_\varepsilon b_h|_0 = \tfrac{1}{2}(I-H_h)(I+H_h)^{-1}a.$$

Here parameter derivatives carry the subscript zero, whereas ordinary derivatives of a are spatial. The derivative of Q_h is $M_{u_h}H_h + H_hM_{u_h}$.

For clarity the differentiation of the inverse metric is made in fixed L^2. The conjugated form of S_f has fixed domain H^1 and depends smoothly on f locally. The addition of the orthogonal projection onto q makes the form coercive by Poincaré, so it can be differentiated and inverted between H^1 and its dual. The derivative of the projection has zero matrix entries between nonconstant base cosine modes. The differentiated form has entries

$$\langle e_n, S_0' e_m\rangle = -\tfrac{1}{4}\int_0^1 a'(e_ne_m)' = -\tfrac{1}{2}\int_0^1 a\,L(e_ne_m).$$

The boundary term vanishes because the derivative of the cosine product is zero at both ends. Thus for $n,m\ge 1$,

$$\langle e_n, G_0' e_m\rangle = \frac{\langle a, L(e_ne_m)\rangle}{4s_ns_m}.$$

For Q_h, subtract its moving rank-one projection before applying the analytic function $z\mapsto hz/(1-z^2)$. The compact Markov operator has a positive gap below its simple fixed eigenvalue at any fixed $h>0$; after projection subtraction the formula for the derivative is the divided difference of this analytic function. Smooth dependence for fixed h follows from that of the scale and the moving projection. At the nonconstant entries this gives

$$\langle e_n, T_h' e_m\rangle = hA(x,y)\langle u_h, e_ne_m\rangle,\quad A(x,y) = \frac{\cosh(x^2/2)+\cosh(y^2/2)}{2\sinh(x^2/2)\sinh(y^2/2)}.$$

Here $x=\pi n\sqrt{h}$ and $y=\pi m\sqrt{h}$. Indeed if $z_j = e^{-hs_j}$, the divided difference multiplied by z_n+z_m is $h(z_n+z_m)(1+z_nz_m)/[(1-z_n^2)(1-z_m^2)]$, equal to $hA(x,y)$.

We estimate a fixed column m. Write $a_j = \int a\cos(\pi jx)\,dx$ and $J_j = (-1)^j a'(1) - a'(0)$. Six integrations by parts give

$$a_j = \frac{J_j}{(\pi j)^2} + \frac{(-1)^j a_3^+ - a_3^-}{(\pi j)^4} + O(j^{-6}),\qquad j\ge 1,$$

where $a_3^{\pm}$ are fixed signed third derivatives. The product $e_n e_m$ is the sum of the cosine modes $|n-m|$ and $n+m$, and the two indices have the same parity. For $n > 2m$, the contribution of the leading coefficients to $h^{-2}\langle e_n, D_h e_m\rangle$ is

$$\frac{J_{n+m}}{2}\,B(x,y), \qquad B(x,y) = A(x,y)[t(x+y)+t(x-y)] - \frac{2}{x^2y^2}.$$

For fixed $x > 0$, as $y \to 0$,

$$B(x,y) \longrightarrow \coth(x^2/4)\,t''(x).$$

To see the limit and obtain uniform bounds, put $X = x^2/2$, $Y = y^2/2$. The identity $t(x)(1+\cosh X)/\sinh X = x^{-2}$ gives the exact decomposition

$$\begin{aligned} B(x,y) = A(x,y)&[t(x+y)+t(x-y)-2t(x)] \\ &+ x^{-2}\big[1/\sinh Y - 2/y^2\big] + t(x)\frac{\cosh Y - 1}{\sinh X \sinh Y}. \end{aligned}$$

For $x \geq 1$ and small y, the first term is $O(x^{-4})$ since $A = O(y^{-2})$ and $t'' = O(x^{-4})$. The other terms are $O(y^2x^{-2})$. At small x, y, $x^2y^2B(x,y)$ has an analytic extension, is even separately, and vanishes on both axes by the same identity. It is therefore divisible by x^2y^2 and B is bounded. Together these facts give the common envelope $|B(x,y)| \leq C\min(1, x^{-2})$, and a locally uniform limit for $x > 0$.

We record why the two smaller coefficient classes do not affect the sharp constant. For $z_{\pm} = x \pm y$ set $B_{\pm} = A(x,y)t(z_{\pm}) - 1/(x^2y^2)$. The fourth-order coefficient contribution, up to its bounded endpoint factors, is

$$h^2\left\{\frac{h}{x^2}B(x,y) + h\sum_{\pm} B_{\pm}(z_{\pm}^{-2} - x^{-2})\right\}.$$

When $x \leq 1$ and $x > 2y$, $|B_{\pm}| \leq C(x/y + y^2/x^2)$. Indeed subtract $t(x)$, use $t'(r) = O(r^3)$ near zero, $A = O((x^2y^2)^{-1})$, and use the last two terms of the preceding decomposition for the remaining half correction. When $x \geq 1$ the analogous bound is $C(x^{-3}/y + y^2/x^2)$. As $|z_{\pm}^{-2} - x^{-2}| \leq Cy/x^3$, the entire fourth-order contribution is bounded by Ch^2/n^2 in both ranges. For the $O(j^{-6})$ coefficients, the derivative multiplier difference before applying a_j is $O_m(h^2n^4)$ if $x \leq 1$, by Taylor expansion of the hyperbolic functions, and $O_m(1)$ if $x \geq 1$, since $j = |n \pm m| \asymp n$. Thus their column norm is $O_m(h^2)$. The finitely many entries $n \leq 2m$ are $O_m(h^2)$ by the same fixed-mode Taylor expansion. Finally, the constant output entry is $O_m(h^2)$: differentiate $(T_h(f_\varepsilon) - G(f_\varepsilon))q_\varepsilon = 0$, and use the $O(h^2)$ fixed-mode constant-density bias.

If η is a finite sum of base cosine modes, the leading output for each parity and each fixed $x > 0$ after dividing by h^2 is therefore

$$\frac{\coth(x^2/4)t''(x)}{2\sqrt{2}}\left[(-1)^n a'(1)\eta(1) - a'(0)\eta(0)\right].$$

The envelope established above is square integrable, and the lower-order columns have norm $O(h^2)$. Sum squares on the two parity sublattices of spacing $2\pi\sqrt{h}$ to obtain Theorem 3. The limiting multiplier tends to $-1/4$ at zero by expansion of t, so the integral is positive. At fixed input and output modes the derivative entries are instead $O(h^2)$. The obstruction comes from output frequencies of scale $h^{-1/2}$, and a fixed cosine-to-cosine contraction need not detect it.

Naturally compatible moving tangents. Let η be a finite mean-zero cosine polynomial as above, and take the moving section

$$\eta_\varepsilon = q_\varepsilon \left[\eta - \int_0^1 f_\varepsilon \eta \right].$$

Then $\eta_\varepsilon \perp q_\varepsilon$ and $(\eta_\varepsilon/q_\varepsilon)' = 0$ at the endpoints for every ε. For the derivative at zero of $(T_h(f_\varepsilon) - G(f_\varepsilon))\eta_\varepsilon$, the same sharp limit as in Theorem 3 holds with profile

$$\coth(r^2/4)t''(r) + 2F(r)/r^2.$$

Indeed, the additional term in the derivative is $(T_h(1) - G(1))(a\eta/2 - \int a\eta)$. The constant is killed. The endpoint slopes of $a\eta/2$ are $a'\eta/2$, since the cosine tangent has zero endpoint derivatives. The constant-density Fourier multiplier therefore gives the stated extra profile, on the same two parity sublattices and with the same boundary factors. The new profile tends to $-1/3$ at zero, so its square integral is positive.

5 The exact profile at a fixed density

Write $\partial_{\mathrm{in}} = -\partial_\nu$ for the inward derivative. Smoothness on the interval does not impose an even-extension condition on the density or tangent.

Theorem 4. *Let $f = q^2$ be smooth and strictly positive, and let η be smooth with $\eta \perp q$. For each endpoint a set $E_a = \partial_{\mathrm{in}}\eta(a)$ and $J_a = \eta(a)\partial_{\mathrm{in}} \log q(a)$. Then*

$$\lim_{h\downarrow 0} h^{-7/2} \|(T_h(f) - G(f))\eta\|_2^2 = \frac{1}{2\pi} \sum_{a=0,1} \int_0^\infty \left[2E_a \frac{F(r)}{r^2} + J_a \coth(r^2/4)t''(r) \right]^2 dr.$$

The limit is positive if and only if one of the pairs (E_a, J_a) is nonzero. In particular, every density with a nonzero endpoint slope has smooth tangents with strong bias of exact order $h^{7/4}$. The same conclusion holds for some tangents satisfying the natural endpoint condition $(\eta/q)' = 0$.

5.1 Localized normalization and residual

Put $\delta = \sqrt{h}$, let z_a be the inward distance from endpoint a divided by δ, and let K be time-one reflecting heat on $[0, \infty)$. Write $D_1 = Kz - z$ and $D_3 = Kz^3 - z^3 - 3z$. Both have Gaussian decay at infinity, by taking the difference between even reflection and the unreflected first and third Gaussian moments. The definitions of t, Ψ give

$$(I + K)\Psi = D_1, \qquad Kz^2 = z^2 + 1.$$

For the first identity, the even extension of D_1 equals the integral of whole-line heat kernels from time zero to one, by $(|z|)'' = 2\delta_0$. Its Fourier transform is $2(1 - e^{-r^2/2})/r^2$, which equals $(1 + e^{-r^2/2})2t(r)$. The even profile Ψ decays faster than any inverse power, by integrating its Fourier formula by parts.

Use $W_M(x) = \sum_a (1 + z_a)^{-M}$. For arbitrarily large fixed M, the following elementary reflecting expansion holds for smooth v:

$$H_h v = v + \tfrac{h}{2} v'' + \tfrac{h^2}{8} v^{(4)} + \sum_a \left[\delta\, \partial_{\mathrm{in}} v(a) D_1(z_a) + \tfrac{\delta^3}{6} \partial_{\mathrm{in}}^3 v(a) D_3(z_a) \right] + O(\delta^5 W_M + \delta^6).$$

To verify it, extend v smoothly past the endpoints. Unreflected heat has the stated second-order expansion by Taylor and Gaussian moments. Compare the reflected input to the smooth extension outside each endpoint, in its inward local coordinate. Their difference is twice the odd Taylor terms through degree three plus $O(|x-a|^5)$. Convolving the linear and cubic odd terms on the outside half-line gives the two displayed defects. The remainder has the factor $\delta^5 W_M$ by Gaussian tails, with remote images smaller than any power of δ. All bounds are uniform for smooth bounded input families.

Every reflected Gaussian kernel and scaled Schwartz kernel used here is bounded on the sup spaces with weight $\delta^j W_M + \delta^k$, with constant independent of h for fixed M, j, k. Indeed the endpoint distances are Lipschitz, so $W_M(y) \le W_M(x)(1+|x-y|/\delta)^M$, and the kernels have uniformly bounded rescaled polynomial moments. The same argument on the doubled interval uses distance on the circle and the reflected endpoint sets. A Lipschitz multiplication commutator then gains a factor δ. We may decrease M by any prescribed fixed number when multiplying a boundary expression by powers of $1 + z_a$.

Put $V = q''/(2q)$, $B_0 = 1 - hV/2 + h^2V^2/8$, and $d_a = \partial_{\mathrm{in}} q(a)$, $c_a = q(a)$. There is a localized scale approximation

$$b_h = B_0 - \delta \sum_a (d_a/c_a)\Psi(z_a) + O(\delta^2 W_M + \delta^6).$$

Here and below a weighted O-bound is pointwise. For the bulk terms, the formal smooth heat expansion in $M_{B_0} H_h M_{B_0}$ is $I - hS + h^2S^2/2$, where $S = L + V$ denotes the interior differential expression. It fixes q to order $O(h^3)$ because $Sq = 0$. The leading reflecting discrepancy in the normalization equation is $\delta \sum_a d_a D_1(z_a)$. The two linear appearances of the proposed correction cancel it by $(I + K)\Psi = D_1$. Replacing $q(x)$ by c_a in these localized terms costs $O(\delta^2 W_M)$, and the layer products and remaining heat defects have that same bound. Comparison of the half-line convolution of these profiles with reflected interval heat costs smaller than any polynomial order, since the profiles are rapidly decaying. Thus the displayed approximation solves the normalization equation up to $O(\delta^2 W_M + \delta^6)$. In the contraction from Section 2, R_h is uniformly bounded in the corresponding weighted sup norm, and the linear commutator and quadratic Lipschitz bounds are $O(\delta)$ there on the ordinary $O(\delta)$ ball. Subtract the approximate fixed-point equation from the exact one and absorb the error term to obtain the bound.

Write $\chi_h = (I + Q_h)^{-1}\eta$. We also have

$$\chi_h = \tfrac{1}{2}\eta + \tfrac{h}{4} S\eta - \tfrac{\delta}{2} \sum_a (E_a - J_a)\Psi(z_a) + O(\delta^2 W_M + \delta^4).$$

In fact the preceding scale approximation and the heat expansion give

$$Q_h\eta = \eta - hS\eta + \delta \sum_a (E_a - J_a) D_1(z_a) + O(\delta^2 W_M + \delta^4).$$

The bulk order is given by B_0; the leading layer is the reflecting defect of η, minus the two normalization factors $J_a(I+K)\Psi$. Apply $I+Q_h$ to the proposed approximation for χ_h. The smooth $hS\eta/4$ term has bulk value $hS\eta/2$ after that operation, with error $O(\delta^2 W_M + \delta^4)$. On the rapidly decaying profiles Q_h differs from K at the rescaled endpoint by $O(\delta W_M)$, using $b_h = 1 + O(\delta)$ and reflected kernel decay. The leading terms cancel, leaving precisely the stated error. Since $Q_h - H_h = O(\delta)$ on the weighted sup spaces, $(I + Q_h)^{-1}$ is bounded there by perturbation of R_h. This proves the approximation.

Set $\psi = S_f^{\dagger}\eta$, $w = \psi/q$, and

$$\mathfrak{r}_h = 2hQ_h\chi_h - (I - Q_h)\psi.$$

The weighted Poisson equation is $\eta = -qw''/2 - q'w'$, with $w' = 0$ at both endpoints. At an endpoint put $u = \partial_{\mathrm{in}}^2 w(a)$, $v = \partial_{\mathrm{in}}^3 w(a)$. Then $-c_a u/2 = \eta(a)$ and $c_a v + 3d_a u = -2E_a$. We claim

$$\mathfrak{r}_h = \delta^3 \sum_a R_a(z_a) + O(\delta^4 W_M + \delta^6), \qquad R_a = E_a(\Psi - D_3/3) + J_a(I+K)(z^2\Psi).$$

For the first term of $\mathfrak{r}_h$, use $2hQ_h\chi_h = 2h(\eta - \chi_h)$. The preceding approximation gives its bulk $h\eta - h^2 S\eta/2$, and layer $\delta^3(E_a - J_a)\Psi$. For the second term use $\ell = b_h q$ and

$$(I - Q_h)\psi = b_h[wH_h\ell - H_h(\ell w)].$$

To bound errors, replacing b_h, ℓ by their localized approximants costs $O(\delta^4 W_M + \delta^6)$. Indeed an inner error always multiplies $w(x) - w(y)$ under heat. For each localized endpoint factor this difference is bounded by $C\delta^2(1 + z_a(x) + z_a(y))^2$ because $w'(a) = 0$. Gaussian weighted bounds absorb the polynomial. A uniform $O(\delta^6)$ error needs no such gain. For the outer error, the same estimate bounds the local difference integral by $C\delta^2(1 + z_a(x))^2$, after Gaussian integration. This proves the error assertion.

The smooth part of the approximant gives the same bulk $h\eta - h^2 S\eta/2$ as the first term. Its reflecting defect at a, by the displayed heat expansion and $w'(a) = 0$, is

$$\delta^3 \left[\frac{d_a u}{2} z^2 D_1 - \frac{c_a v + 3d_a u}{6} D_3 \right](z_a).$$

Indeed the two linear defects have coefficients differing by $d_a(w(x) - w(a))$; their first nonzero difference is $d_a\delta^2 u z^2/2$. The cubic coefficients differ at the endpoint by $-c_a v - 3d_a u$. Replacing ℓ in the inner term by its boundary layer adds $-\delta^3 d_a u[z^2 K\Psi - K(z^2\Psi)]/2$ at a. This follows by multiplying the layer $-\delta d_a \Psi$ by the quadratic difference of w. The outer layer acting on the leading bulk adds $-\delta^3 J_a\Psi$. All higher products and Taylor remainders are $O(\delta^4 W_M + \delta^6)$ by the same polynomial distance estimates. Use $D_1 = \Psi + K\Psi$ and the two Poisson identities to simplify the layer of $(I - Q_h)\psi$ to

$$\delta^3 \left[(E_a/3)D_3 - J_a\big((I+K)(z^2\Psi) + \Psi\big) \right](z_a).$$

Subtracting proves the residual formula.

5.2 A Neumann inverse bound

Let $P = q^{-1}Q_h q$, acting reversibly on $L^2(f\,dx)$, and let $H_N^2 = \{u \in H^2 : u'(0) = u'(1) = 0\}$. For g of zero f-mean,

$$\|(I - P)^{-1} g\|_{L^2(f)} \le C \left(\|g\|_{L^2(f)} + h^{-1} \|fg\|_{(H_N^2)^*} \right).$$

Here the dual norm uses unweighted integration on the interval. To prove it, $Pu = H_h(\ell u)/(H_h\ell)$ with $\ell = b_h q$, and the Dirichlet form of $I - P$ is

$$\tfrac{1}{2} \iint H_h(x,y)\ell(x)\ell(y)|u(x) - u(y)|^2 \, dx\, dy.$$

Uniform positive upper and lower bounds on ℓ compare this form to that of $I - H_h$. The Neumann spectral gap and equivalence of weighted and unweighted centered variances give $\|(I - P)^{-1}\|_{L_0^2(f) \to L_0^2(f)} \le C/h$. Also $\ell = q^2/(H_h\ell)$ is uniformly Lipschitz. Indeed $\|\ell - q\|_\infty = O(\delta)$, so $\|\partial_x H_h(\ell - q)\|_\infty = O(1)$ by the Gaussian derivative bound, and $H_h q$ is uniformly Lipschitz on the doubled circle. The same argument or Gaussian cancellation gives $\|\partial_x^2 H_h \ell\|_\infty = O(\delta^{-1})$.

Put $y = P(I-P)^{-1}g$, so $y = P(y+g)$ and y is smooth with Neumann derivative. The equation gives

$$(I - H_h)y = (H_h\ell)^{-1}\{H_h(\ell y) - (H_h\ell)(H_h y) + H_h(\ell g)\}.$$

The commutator in braces is the covariance double integral of the increments of ℓ, y with two copies of the heat kernel, divided by two. On the doubled circle its L^2 norm plus h times the L^2 norm of its second derivative is at most $Ch\|y'\|_2$. For each derivative of the kernels, use a Gaussian bound at twice the variance with factor $h^{-j/2}$. The two translation increments have product L^2 norm at most $C|u-v|^2\|y'\|_2$ for shifts u, v; integration gives the factor h. The smoothed source $H_h(\ell g)$ has the same scaled norm bounded by $C\|g\|_2$, by its Fourier multiplier. Multiplication by $(H_h\ell)^{-1}$ preserves that scaled norm up to a constant, by its first and second derivative bounds and the Fourier inequality $\sqrt{h}\|u'\|_2 \lesssim \|u\|_2 + h\|u''\|_2$.

The Fourier inequality

$$h\|y''\|_2 \le C\big(\|(I - H_h)y\|_2 + h\|((I - H_h)y)''\|_2\big)$$

follows from $u \lesssim (1+u)(1-e^{-u})$ for $u \ge 0$. The preceding bounds give $h\|y''\|_2 \le C\|g\|_2 + Ch\|y'\|_2$. Absorb the first derivative by interpolation and use the gap bound for $h\|y\|_2$. This proves a uniform $L^2_0(f) \to H^2_N$ bound for $hP(I-P)^{-1}$. The operator is selfadjoint on the weighted space. Duality, subtracting the weighted mean of each dual test before applying the bound, gives the claimed inverse estimate from $(I-P)^{-1} = I + P(I-P)^{-1}$.

5.3 Recovering the boundary layer

Let U_a be the decaying solution of $(I-K)U_a = R_a$, defined by even Fourier transform. For $r \ne 0$, write $e = e^{-r^2/2}$. The even transform of $\Psi - D_3/3$ equals $-2/r^2 + 2t(r) + 4(1-e)/r^4$. This follows by transforming the even powers $|z|$ and $|z|^3$, whose second and fourth derivatives are $2\delta_0$ and $12\delta_0$, respectively, and by using that the localized profile is integrable. The transform is continuous at zero. For the second profile the transform is $-2(1+e)t''(r)$. Both numerators vanish to order at least two at zero, so after division by $1-e$ the even extension of U_a has Fourier transform

$$-4E_aF(r)/r^2 - 2J_a\coth(r^2/4)t''(r).$$

The singularities at zero cancel, and the multiplier and all its derivatives are integrable at infinity. Thus U_a decays rapidly by integration by parts. Put $v_h = \delta^3\sum_a U_a(z_a)$. Reflecting heat and the scale bound give

$$(I - Q_h)v_h = \delta^3\sum_a R_a(z_a) + O(\delta^4 W_M + \delta^6).$$

Indeed the difference from the half-line profile is smaller than any power due to the distant-endpoint tails; replacing either b_h by one costs $O(\delta^4 W_M)$. The projection of v_h onto q is $O(\delta^4)$ in L^2, by the integral of its decaying profiles.

Set $e_h = \mathfrak{r}_h - (I - Q_h)v_h$. In addition to the pointwise bound $O(\delta^4 W_M + \delta^6)$, this error has $\int_I qe_h = O(\delta^6)$ on either half interval I. We give the mass argument, since a global zero mean alone does not suffice for the inverse estimate. For any smooth or bounded u, integration of

$$q(I - Q_h)(qu) = \ell[(H_h\ell)u - H_h(\ell u)]$$

over a set I is exactly its cross-cut integral

$$\int_{x\in I,\ y\notin I} H_h(x,y)\ell(x)\ell(y)[u(x) - u(y)]\,dx\,dy.$$

In particular the cross-cut integrals for v_h at the interior midpoint are smaller than every power of δ. Use $\mathfrak{r}_h = h\eta - h(I-Q_h)\chi_h - (I-Q_h)\psi$. On the left half interval $\int q\eta = -f(1/2)w'(1/2)/2$. Compare its two cross-cut integrals at $1/2$ with a smooth auxiliary periodic model whose q, w agree in a fixed neighborhood of that cut. Choose the auxiliary w to be constant near a second cut to its left and use the arc between those cuts. Then the auxiliary $\eta = S(qw)$ gives the same differential flux integral over the arc. The bulk scale approximations make the two b_h's differ by $O(\delta^6)$ in the matched neighborhood. The two χ_h's differ by $O(\delta^4)$ there, since their first two interior terms agree. The heat kernel at an interior cut is the same unreflected Gaussian up to superpolynomial error. Its cross-cut mass is $O(\delta)$, and the contribution outside the matched neighborhood is superpolynomial. Thus the flux for $\psi = qw$ matches to $O(\delta^6)$, and the flux for χ_h matches to $O(\delta^6)$ after its factor h. At the auxiliary second cut the w flux is superpolynomial, and the χ_h flux after multiplication by h is $O(\delta^7)$, since locally $\eta = 0$ and $\chi_h = O(\delta^4)$. The auxiliary periodic residual is $O(h^3)$: the pure smooth expansions of Q_h through h^2 and of χ_h through h give cancellation of the bulk h and h^2 terms above. Therefore the original half-mass is $O(\delta^6)$. Reflection proves the same bound on the other half.

For any $\varphi \in H^2_N$, subtract $\varphi(a)$ on the half interval adjacent to endpoint a. The bound $|\varphi(x) - \varphi(a)| \le C\,\mathrm{dist}(x,a)^{3/2}\|\varphi''\|_2$ uses the zero endpoint derivative. The per-half masses and the pointwise error give $\|qe_h\|_{(H^2_N)^*} = O(\delta^6 + \delta^{13/2})$. Also $\|e_h\|_2 = O(\delta^{9/2} + \delta^6)$. Apply the inverse bound to e_h/q, which has zero f-mean. Since $2(I-Q_h)(T_h - G)\eta = \mathfrak{r}_h$, we conclude

$$\|2(T_h - G)\eta - v_h\|_2 = O(\delta^4) = o(\delta^{7/2}).$$

Here we included the $O(\delta^4)$ projection of v_h. The two endpoints are separated on the scaled coordinate, so their rapidly decaying profiles have negligible cross term after dividing the squared norm by δ^7. Even Fourier Plancherel gives the stated integral. The two multiplier shapes have different ratios at zero and infinity: the ratio of $\coth(r^2/4)t''(r)$ to $F(r)/r^2$ tends to 6 and -6, respectively. Thus their linear combination vanishes only when both coefficients vanish. At a nonzero slope, take a tangent nonzero near that endpoint and subtract an interior bump for orthogonality. For a naturally compatible tangent take q times a smooth function constant and nonzero near that endpoint and remove its weighted mean by an interior bump. Both choices give a nonzero J_a.

6 Weak bias and the Fisher Hessian

The previous strong limit need not be visible in a contraction against a fixed smooth tangent. Let $w_\eta = S_f^\dagger \eta/q$ and

$$\mathfrak{s}(\zeta,\eta) = \tfrac{1}{2}\int_0^1 f(\zeta/q)'(\eta/q)'.$$

This is the closed form of S_f on these smooth tangents.

Theorem 5. *For fixed smooth $\zeta, \eta \perp q$,*

$$\lim_{h\downarrow 0} h^{-2}\langle \zeta, (T_h(f) - G(f))\eta\rangle = \tfrac{1}{24}\int_0^1 fV''w'_\zeta w'_\eta - \tfrac{1}{12}\mathfrak{s}(\zeta,\eta) + \tfrac{1}{6}\sum_{a=0,1} \partial_{\mathrm{in}} \log q(a)\,\zeta(a)\eta(a).$$

For a diagonal contraction this is

$$-\tfrac{1}{48}\,\mathrm{Hess}_{W_2} I(f)[q\eta, q\eta], \qquad I(f) = \int_0^1 f|(\log f)'|^2,$$

where the Hessian is the second derivative along the one-dimensional smooth quadratic Wasserstein geodesic. Thus the periodic Fisher identity survives in weak form, with the natural interval geodesic and its boundary term.

For each input denote its residual from Section 5 by $\mathfrak{r}_{\eta,h}$, and its cumulative weighted residual by

$$\mathcal{F}_{\eta,h}(x) = \int_0^x q\mathfrak{r}_{\eta,h}.$$

It vanishes at both endpoints because the residual is orthogonal to q. For each fixed x in the open interval,

$$h^{-3}\mathcal{F}_{\eta,h}(x) \longrightarrow \frac{f(x)}{12}\left[(S\eta/q)'(x) - V''(x)w'_\eta(x)\right].$$

To see this, use the periodic cross-cut comparison from Section 5 at the arbitrary fixed interior point x in place of the midpoint. The flux discrepancy is $o(h^3)$, since the bulk scales near the cut agree to $O(\delta^6)$ and the χ_h's agree to $O(\delta^4)$, their respective cut integrals having mass $O(\delta)$, and the latter flux having prefactor h. At the auxiliary cut the w flux is superpolynomial and the $h\chi_h$ flux is $O(\delta^7)$. For the smooth periodic model, Lemma 3.2 of [1] gives, on fixed smooth inputs,

$$Q_h = I - hS + h^2S^2/2 + h^3(-S^3/6 + R) + o(h^3).$$

In density coordinates its operator R satisfies

$$q^{-1}R(qw) = -[fV''w']'/(12f).$$

Expanding $I + Q_h$ on smooth input also gives $\chi_h = \eta/2 + hS\eta/4 + O(h^3)$ in L^2; the quadratic coefficient cancels, and the norm of $(I + Q_h)^{-1}$ is bounded. Consequently the periodic residual divided by h^3 tends to $R(qw_\eta) - S^2\eta/6$. Multiplying this expression by q makes it a derivative with primitive $f[(S\eta/q)' - V''w'_\eta]/12$. At the auxiliary second cut this primitive is zero. This proves the pointwise limit with the stated sign.

Let $J^{\mathrm{int}}_{\eta,a}(z) = \int_0^z R_a(t)\,dt$ for the two layer profiles of Section 5. They are rapidly decaying because R_a is rapidly decaying and has zero mean, as its Fourier transform vanishes at zero. On the left half interval, subtract $\delta^4 q(0)J^{\mathrm{int}}_{\eta,0}(x/\delta)$ from $\mathcal{F}_{\eta,h}(x)$. The resulting error is bounded by

$$C\big(\delta^5(1 + x/\delta)^{-M} + \delta^6\big)$$

for each fixed M, with its own constant. Indeed start from the $O(\delta^6)$ signed mass at the midpoint and integrate the pointwise residual estimate backwards to x. After the leading profile, the localized integrand is $O(\delta^4 W_{M+1})$, also accounting for $q(x) - q(0)$ on the profile scale. Its remaining tail gives the displayed bound. The right half interval has the reflected envelope and the signed profile $-\delta^4 q(1)J^{\mathrm{int}}_{\eta,1}((1 - x)/\delta)$. These signs follow from the zero total mass.

Integrating by parts gives $\langle \psi_\zeta, \mathfrak{r}_{\eta,h}\rangle = -\int w'_\zeta \mathcal{F}_{\eta,h}$, where $\psi_\zeta = S_f^\dagger \zeta$. After removing the cumulative profiles, the normalized integrand converges to its bulk expression by dominated convergence up to a negligible localized error. To justify the latter statement, w'_ζ vanishes at the endpoints, so the $\delta^5 W_M$ envelope contributes $O(\delta^7)$ to the integral. The remaining bound is $O(\delta^6)$ uniformly, and the pointwise limit in the open interval applies; equivalently, first integrate on compact interior subintervals, then use this uniform bound and the localized estimate on the complementary collars. The cumulative profiles themselves contribute in the normalized limit

$$\sum_a \frac{q(a)\partial^2_{\mathrm{in}} w_\zeta(a)}{2} \int_0^\infty z^2 R_a(z)\,dz.$$

This follows by rescaling $w'_\zeta = \pm\delta z\partial^2_{\mathrm{in}} w_\zeta(a) + O(\delta^2 z^2)$ at the two ends and using $\int_0^\infty z J^{\mathrm{int}}_{\eta,a}(z)\,dz = -\frac{1}{2}\int_0^\infty z^2 R_a(z)\,dz$.

Finally, the exact inverse relation on $q^\perp$ gives

$$\begin{aligned} 2\langle \zeta, (T_h - G)\eta\rangle = h^{-1}\langle \psi_\zeta, \mathfrak{r}_{\eta,h}\rangle + \langle \chi_{\zeta,h}, \mathfrak{r}_{\eta,h}\rangle \\ + \tfrac{2}{h}\langle (T_h - G)\zeta, \mathfrak{r}_{\eta,h}\rangle. \end{aligned}$$

The last term is $O(\delta^5)$ by the strong estimate and the residual L^2 bound. The middle term is $O(\delta^5)$ as well: $\chi_{\zeta,h} = \zeta/2 + O(\delta)$ uniformly, the residual has $O(\delta^4)$ absolute integral, and its leading localized profile has zero integral at each endpoint. Its contraction with ζ therefore gains another factor of δ. It remains to evaluate half the previously computed limit $h^{-3}\langle \psi_\zeta, \mathfrak{r}_{\eta,h}\rangle$.

The expansion at frequency zero of $\widehat{R}_a$ from Section 5 is $r^2(E_a/12 + J_a/4) + O(r^4)$. Hence $\int_0^\infty z^2 R_a = -E_a/12 - J_a/4$. Since $q(a)\partial^2_{\mathrm{in}} w_\zeta(a) = -2\zeta(a)$, the boundary contribution after taking half is $\sum_a \zeta(a)(E_a/24 + J_a/8)$. Integrating the bulk term by parts, with $w'_\zeta = 0$ at the two endpoints, gives $\frac{1}{24}\int f V'' w'_\zeta w'_\eta - \frac{1}{12}\int \zeta S\eta$. The latter integral is the interior differential expression. A further integration by parts gives

$$\int_0^1 \zeta S\eta = \mathfrak{s}(\zeta,\eta) + \tfrac{1}{2}\sum_a \zeta(a)(E_a - J_a).$$

The E_a's cancel from the weak coefficient and the J_a's add to $\frac{1}{6}\sum_a \zeta(a)J_a$. This proves the bilinear assertion.

To identify the Fisher Hessian, write $\ell = \log f$ and $v = w'_\eta/2$. Then $q\eta = -(fv)'$, $v = 0$ at both endpoints, and $x \mapsto x + tv(x)$ is the increasing Wasserstein displacement for small t. It fixes the two endpoints and leaves a smooth positive density on the interval. The change of variables formula gives

$$I(f_t) = \int_0^1 f\left[\frac{\ell'}{1+tv'} - \frac{tv''}{(1+tv')^2}\right]^2.$$

Differentiating twice and integrating the cross term yields

$$I''(0) = 2\int_0^1 f[(v'')^2 - 3\ell''(v')^2] + 6[f\ell'(v')^2]_0^1.$$

Also $\eta/q = -v' - \ell' v$ and $4V'' = \ell'''' + \ell'\ell''' + (\ell'')^2$. Expanding the square and integrating by parts using $v(0) = v(1) = 0$ gives

$$\int_0^1 f[(v'' + \ell' v' + \ell'' v)^2 - 4V'' v^2] = \int_0^1 f[(v'')^2 - 3\ell''(v')^2] + [f\ell'(v')^2]_0^1.$$

At an endpoint $f(v')^2 = \eta^2$, so the boundary bracket is $-2\sum_a \eta(a)J_a$. Substitute these identities into the diagonal weak coefficient to obtain $-I''(0)/48$.

7 An optimal form-comparable correction at uniform density

The compact positive corrections in the periodic companion preserve the original tensor form up to uniform two-sided comparison. We show what can be attained under the same requirement for ordinary H^4 tangents on the interval, without imposing an endpoint condition on their derivatives.

Theorem 6. *Work at $f = 1$ and on the mean-zero subspace. If $0 \le C_h \le MT_h(1)$ for a constant M independent of h, then for any fixed smooth mean-zero η with a nonzero endpoint derivative,*

$$\|(C_h - G(1))\eta\|_2 \ \gtrsim_\eta \ (h/\log(1/h))^{7/4}.$$

If each C_h is diagonal in the Neumann basis, the lower bound strengthens to $h^{7/4}$. There exist compact positive corrections with

$$cT_h(1) \le C_h \le CT_h(1), \qquad \|C_h - G(1)\|_{H^4\cap 1^\perp \to L^2} \ \lesssim \ (h/\log(1/h))^{7/4}.$$

They can be formed from a diagonal comparison and a selfadjoint correction of rank at most four.

For the first lower bound put P_N for the orthogonal projection onto $e_n,\ n > N$. For any L^2 input, Cauchy–Schwarz for the positive form gives

$$\|P_N C_h \eta\|_2 \le M\sqrt{\|P_N T_h(1) P_N\|}\sqrt{\langle \eta, T_h(1)\eta\rangle}.$$

Here the second factor is bounded, since $T_h(1) \le G(1)$. Choose $N \asymp \sqrt{\log(1/h)/h}$ so that $hs_N \ge 4\log(1/h)$. The first factor is then $O(h^{5/2})$, by the exact heat-tensor eigenvalues. In contrast, the cosine coefficient asymptotic in Theorem 2 and $G(1)e_n = (\pi n)^{-2}e_n$ give $\|P_N G(1)\eta\|_2 \asymp_\eta N^{-7/2}$. At least one of the parity coefficients is nonzero. This yields the lower bound by the triangle inequality. If C_h is diagonal, choose $N \asymp h^{-1/2}$ with a large fixed proportionality constant so that $Mh/(2\sinh(hs_n)) \le (\pi n)^{-2}/2$ for $n > N$. Then the coordinatewise error is at least half the corresponding $G(1)$ tail, giving the strengthened bound.

For the construction abbreviate the eigenvalues by $T_n = h/(2\sinh(hs_n))$ and $G_n = (\pi n)^{-2}$. Define a base diagonal operator A_h to have eigenvalue G_n when $n \le J$, where J is the largest integer with $hs_J \le 1$, and T_n otherwise. Then $T_h(1) \le A_h \le (\sinh 1)T_h(1)$. Take N to be the largest integer with $hs_N \le \log(1/h)$. For each parity p choose a positive k_p of that parity with $k_p \asymp h^{-1/8}$. For small h these satisfy $k_p \le J < N$. Set $w_p = (\pi k_p)^2 e_{k_p}/\sqrt{2}$ and define v_p by its coefficients

$$(v_p)_n = \frac{\sqrt{2}(G_n - T_n)}{(\pi n)^2}\mathbf{1}_{\{J<n\le N,\ n \text{ of parity } p\}}.$$

Use

$$C_h = A_h + \sum_p (v_p \otimes w_p + w_p \otimes v_p).$$

All corrections are selfadjoint, and A_h is a finite modification of the compact operator $T_h(1)$. The square-root inverse norms obey

$$\|T_h(1)^{-1/2}w_p\| = O(h^{-3/8}), \qquad \|T_h(1)^{-1/2}v_p\|^2 \le Ch^{-1}e^{hs_N}\sum_{n>J} n^{-8} = O(h^{3/2}).$$

For the first bound $hs_{k_p} \to 0$, so $T_{k_p} \asymp k_p^{-2}$. For the second use $0 \le G_n - T_n \le G_n$. Thus the perturbation has form bound $O(h^{3/8})T_h(1)$. This proves positivity, compactness, and uniform two-sided comparison for small h.

For $\eta \in H^4$ the cosine coefficient expansion used above has $O(n^{-4}\|\eta\|_{H^4})$ remainder by integration by parts, the endpoint trace bounds, and Cauchy–Schwarz. For each parity put $\ell_p(\eta) = (-1)^p\eta'(1) - \eta'(0)$. The low probe gives $\langle w_p, \eta\rangle = \ell_p(\eta) + O(k_p^{-2}\|\eta\|_{H^4})$. On $J < n \le N$ the correction therefore replaces the defect $(G_n - T_n)\langle e_n, \eta\rangle$ up to an error bounded in norm by $C(J^{-11/2} + k_p^{-2}J^{-7/2})\|\eta\|_{H^4}$, using $G_n - T_n \le G_n$. The reciprocal term $w_p\langle v_p, \eta\rangle$ at the low probe contributes at most $Ck_p^2\sum_{n>J} n^{-6}\|\eta\|_{H^4} = O(h^{9/4}\|\eta\|_{H^4})$. The uncorrected tail beyond N is $O(N^{-7/2}\|\eta\|_{H^4})$ and dominates these smaller errors. This proves the matching upper bound. As in the periodic setting, the form comparison preserves both the completed form space and the square-root range, the latter by Douglas's range theorem [6].

7.1 Form-comparable correction at fixed densities

The exact boundary profile also permits a sharp correction at every fixed smooth positive density. No commutation with the Neumann basis is required.

Theorem 7. *Fix a smooth positive density f. On $q^\perp$ there exist compact positive C_h with $cT_h(f) \le C_h \le CT_h(f)$ and*

$$\|C_h - G(f)\|_{H^{10}\cap q^\perp \to L^2} \lesssim_f (h/\log(1/h))^{7/4}.$$

One can take $C_h - T_h(f)$ of rank at most eight. For any uniformly dominated positive correction $0 \le C_h \le MT_h(f)$, and any fixed smooth tangent η for which $(S_f^\dagger \eta/q)'''$ is nonzero at an endpoint,

$$\|(C_h - G(f))\eta\|_2 \gtrsim_{f,\eta,M} (h/\log(1/h))^{7/4}.$$

In particular the operator rate is optimal under uniform form domination.

We retain f fixed and use the bounds of Section 5 uniformly in η, with constants times $\|\eta\|_{H^{10}}$. This follows from the one-dimensional Sobolev and trace bounds and the Poisson equation for w_η, which gives $w_\eta \in H^{12}$ with its norm bounded by $C_f\|\eta\|_{H^{10}}$ after fixing the weighted mean. The localized expansion uses at most six derivatives of η or qw_η and a bounded number of smoother coefficients depending only on f. In the auxiliary periodic comparison the smooth extensions near the fixed midpoint may be chosen by a fixed cutoff and multiplication, with these same bounds. All steps leading to the $O(h^2)$ strong remainder are linear in the input and satisfy these estimates.

There are four leading profile terms in Section 5, one for each endpoint and each of E_a, J_a. Write their sum for $(T_h - G)\eta$ as $\sum_{j=1}^4 V_j \mathcal{L}_j(\eta)$, with $\mathcal{L}_j$ the four indicated endpoint functionals and V_j the corresponding profiles of amplitude $\delta^3/2$. We may use the even, period-two periodizations of the profiles centered at their respective endpoints, with $O(\delta^M)$ change in L^2 for every fixed M. Then $\|V_j\|_2 = O(\delta^{7/2})$, $\|V_j\|_1 = O(\delta^4)$, and their cosine coefficients are bounded by $C\delta^4(1+\delta n)^{-4}$. The last statement follows from their explicit Fourier transforms in Section 5. Let N be the largest integer with $hs_N \le \log(1/h)$, and let $P_{\le N}$ be the Neumann projection through frequency N, including the constant. Write $\Pi^\perp$ for the projection onto $q^\perp$. Set $v_j = \Pi^\perp b_h P_{\le N} V_j$. Then

$$\|v_j - V_j\|_2 \lesssim (h/\log(1/h))^{7/4}.$$

Indeed the high cosine tail has this bound by summing the fourth-power coefficient decay. Multiplying by $b_h = 1 + O(\delta)$ adds only $O(\delta^{9/2})$. The removed q-component is $O(\delta^4)$: $\langle q, b_h P_{\le N} V_j\rangle = \langle P_{\le N}(qb_h), V_j\rangle$, and the cosine partial sums of $qb_h = \ell$ are uniformly bounded in sup norm. That bound follows by Cauchy–Schwarz from the uniform H^1 norm of ℓ in Section 5. The same reasoning yields

$$|\langle v_j, \eta\rangle| \le C\delta^4\|\eta\|_{H^{10}} \quad (\eta \perp q),$$

since $b_h\eta$ has uniformly bounded H^1 norm in terms of the input norm.

Use a probe frequency $\kappa \asymp h^{-1/16}$. For $\eta(a)$ the cosine projection kernel through frequency κ, evaluated at a, recovers the value to error $O(\kappa^{-1}\|\eta\|_{H^{10}})$; this follows by summing its n^{-2} coefficient bound. For the two inward derivatives, take the even and odd probes $(\pi k)^2 e_k/\sqrt{2}$ with $k \asymp \kappa$ of the two parities, and take their fixed linear combinations to isolate the two endpoints. The boundary expansion for cosine coefficients gives error $O(\kappa^{-2}\|\eta\|_{H^{10}})$. Include the factor $\partial_{\mathrm{in}} \log q(a)$ in the value probes for J_a. Let p_j be the resulting probes and $w_j = \Pi^\perp b_h p_j$. Their norms are $O(\kappa^2)$, all frequencies are $O(\kappa)$, and

$$|\langle w_j, \eta\rangle - \mathcal{L}_j(\eta)| \le C(\kappa^{-1} + \delta\kappa^2)\|\eta\|_{H^{10}},$$

using $b_h = 1 + O(\delta)$ and $\eta \perp q$.

Define $C_h = T_h(f) - \sum_j (v_j \otimes w_j + w_j \otimes v_j)$. For any trigonometric polynomial p, on the tangent space,

$$|\langle \Pi^\perp b_h p, \xi \rangle| \le \|H_h^{-1/2} p\|_2 \, h^{-1/2} \langle \xi, T_h(f) \xi \rangle^{1/2}.$$

This follows from $Q_h = b_h H_h b_h$ and $T_h(f) \ge h Q_h$ as forms on $q^\perp$. For v_j the factor on the right is $O(h^{-1/2} e^{h s_N/2} h^{7/4}) = O(h^{3/4})$. For w_j it is $O(h^{-1/2}\kappa^2) = O(h^{-5/8})$. Their products give a relative form perturbation $O(h^{1/8})$. This gives compactness, positivity, and uniform two-sided comparison for all small h. The outputs $v_j \langle w_j, \eta \rangle$ cancel the leading profile up to the Fourier tail, the smaller probe errors, and the $O(h^2)$ strong remainder. The reciprocal outputs $w_j \langle v_j, \eta \rangle$ cost $O(\kappa^2 \delta^4 \|\eta\|_{H^{10}}) = O(h^{15/8} \|\eta\|_{H^{10}})$. All these errors apart from the Fourier tail are $o((h/\log(1/h))^{7/4})$. This proves the upper bound.

For the lower bound the order-h gap of $I - Q_h$ on $q^\perp$ from Section 5 implies $T_h(f) \le C Q_h$ as forms, extending $T_h(f)$ by zero on q. Thus for any high-mode $g = P_{>N} g$, $\langle b_h^{-1} g, C_h b_h^{-1} g \rangle \le C \langle g, H_h g \rangle$. Form Cauchy–Schwarz and the uniform bound on $\|T_h(f)\|$ give

$$\|P_{>N} b_h^{-1} C_h \eta\|_2 \le C_\eta \exp(-h s_N/2).$$

Here take $N \asymp \sqrt{\log(1/h)/h}$ with a sufficiently large fixed proportionality factor.

In contrast $b_h^{-1} G(f) \eta = (H_h \ell) w_\eta / 2$. Use exponential Fourier coefficients of the even extensions to the doubled circle. For $w = w_\eta$ we have $w'(0) = w'(1) = 0$ and

$$w_n = \frac{w'''(0) - (-1)^n w'''(1)}{(\pi n)^4} + O(n^{-6}), \qquad n > 0.$$

Put $A = H_h \ell$. Its coefficients are $A_m = e^{-h s_{|m|}} \ell_m$. The uniform H^1 bound for ℓ gives $\sum_m |\ell_m| \le C$ and $\sum_m |m A_m| \le C \delta^{-1/2}$, the latter by Cauchy–Schwarz with the Gaussian multipliers. In the Fourier convolution of A and w, for $|m| \le n/2$ replace $(n-m)^{-4}$ by n^{-4}; the summed error is $O(n^{-5} \delta^{-1/2})$. For the complementary indices the sum is $O(e^{-h s_{\lfloor n/2 \rfloor}})$ by boundedness of the w coefficients and summability of ℓ_m. For sufficiently large proportionality factor in N this is $o(n^{-4})$ uniformly on $n > N$. Summing the constant and parity coefficients of A therefore gives

$$(Aw)_n = \frac{A(0) w'''(0) - (-1)^n A(1) w'''(1)}{(\pi n)^4} + o(n^{-4}) \quad \text{uniformly on } n > N.$$

Since $A(a) = q(a)/b_h(a) \to q(a) > 0$, at least one parity has a nonzero limiting coefficient when the stated third derivative condition holds. The norm of the $G(f)$ tail after multiplication by b_h^{-1} is of order $N^{-7/2}$, dominating the C_h tail for the chosen N. Uniform invertibility of the multiplication yields the claimed lower bound. There exist tangents satisfying the third derivative condition, since one can choose smooth w with zero first but nonzero third endpoint derivative and take $\eta = S_f(qw)$, which is automatically orthogonal to q.

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