Bound states of ruled strips with diverging twisting: a Hardy inequality and an exact existence criterion

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Abstract

Krejčiřík and Tiedra de Aldecoa (Annales Henri Poincaré, 2018) studied the Dirichlet Laplacian on the ruled strip swept by a segment (a_1, a_2) rotating about a line with diverging angular velocity; its essential spectrum starts at the lowest Dirichlet eigenvalue Λ_1 of the annulus, or disc, with radii $|a_1|$ and $|a_2|$. They asked whether discrete eigenvalues exist when $a_1 a_2 > 0$, and conjectured that none exist when $a_1 a_2$ is large. We prove that for every $a_1 a_2 > 0$ and every locally Lipschitz twist the operator is bounded below by Λ_1 plus an explicit Hardy weight. Hence, under diverging twisting, the discrete spectrum is empty and Λ_1 is not an eigenvalue: the question has a negative answer and the conjecture holds for all $a_1 a_2 > 0$. The proof compares the transverse potential pointwise with the inverse-square potential of the annulus. For segments meeting the axis, a superposition of cone weights shows that the transverse energy never falls below the smaller of Λ_1 and the flat-strip threshold E_1. With a Bessel ground-state identity for large twisting, this gives an exact criterion: some diverging twist produces discrete eigenvalues below Λ_1 if and only if $E_1 < \Lambda_1$, and every diverging twist does so if and only if $a_1 = -a_2$. Consequently the existence theorems of the same paper, stated for all $a_1 a_2 \le 0$, hold exactly in the symmetric case; for $a_1 = 0$ no twist produces spectrum below Λ_1. For every asymmetric segment and diverging twist the spectrum below Λ_1 is finite.

Keywords. Dirichlet Laplacian; ruled strip; twisted waveguide; diverging twisting; Hardy inequality; discrete spectrum; Bessel functions.
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1 Introduction

1.1 The model and the question

Let $a_1 < a_2$ be real numbers, put $\omega := (a_1, a_2)$, and let $\theta : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz. Krejčiřík and Tiedra de Aldecoa [6] consider the ruled strip

$$\Omega = \big\{(s,\, t\cos\theta(s),\, t\sin\theta(s)) : s \in \mathbb{R},\ t \in \omega\big\} \subset \mathbb{R}^3,$$

swept out by the segment ω rotating about the s-axis with angle $\theta(s)$, and the Dirichlet Laplace–Beltrami operator $-\Delta_D^\Omega$ on it. In the Fermi coordinates (s,t) the metric is $\mathrm{diag}(f^2, 1)$ with

$$f(s,t) := \sqrt{1 + \theta'(s)^2 t^2}, \tag{1.1}$$

and $-\Delta_D^\Omega$ is unitarily equivalent to the self-adjoint operator H in $\mathcal{H} := L^2(\mathbb{R}\times\omega,\, f\,\mathrm{d}s\,\mathrm{d}t)$ associated with the closure h of the form

$$h[\psi] = \int_{\mathbb{R}\times\omega} \Big(\frac{|\partial_s\psi|^2}{f} + f\,|\partial_t\psi|^2 \Big)\,\mathrm{d}s\,\mathrm{d}t, \qquad \psi \in C_0^\infty(\mathbb{R}\times\omega) \tag{1.2}$$

[6, Section 2.1]. Their main regime is *diverging twisting*,

$$\lim_{|s|\to\infty} |\theta'(s)| = \infty \tag{1.3}$$

(an essential limit, θ' being defined almost everywhere). Following [6, (4)], let $r_1 := \min\{|a_1|,|a_2|\}$ if $a_1a_2 \geq 0$, $r_1 := 0$ if $a_1a_2 \leq 0$, and $r_2 := \max\{|a_1|,|a_2|\}$; let Λ_1 be the lowest Dirichlet eigenvalue of the annulus $\{x\in\mathbb{R}^2 : r_1 < |x| < r_2\}$ (a disc when $r_1 = 0$, and then $\Lambda_1 = j_{0,1}^2/r_2^2$ with $j_{0,1}\approx 2.4048$ the first zero of J_0), and let $E_1 := \pi^2/(a_2-a_1)^2$ be the lowest Dirichlet eigenvalue of ω.

The strip looks at infinity like a three-dimensional tube with annular cross-section, and [6, Theorem 1] states $\sigma_{\mathrm{ess}}(H) = [\Lambda_1,\infty)$ under (1.3); we use this statement only for $a_1a_2 > 0$ (Remark 3.4). For the discrete spectrum, [6, Theorem 3] states that $\inf\sigma(H) < \Lambda_1$ for every θ whenever $a_1a_2 \leq 0$, and [6, Theorem 2] states, as its consequence, that $\sigma_{\mathrm{disc}}(H)\cap(0,\Lambda_1) \neq \emptyset$ whenever (1.3) holds and $a_1a_2 \leq 0$. Right after Theorem 2, in Section 1.2 of [6], the authors pose the question of showing that discrete eigenvalues also exist in some cases $a_1a_2 > 0$, and add: "we conjecture that the discrete spectrum is empty whenever a_1a_2 is large enough." Krejčiřík and Zahradová [7, Open Problem 5.7] return to these strips and point to the accumulation of their discrete eigenvalues. Theorem and equation numbers of [6] refer to arXiv:1802.00321v1, the only arXiv version.

This paper answers the question and proves the conjecture for every $a_1a_2 > 0$, and it determines exactly when discrete eigenvalues below Λ_1 exist for segments meeting the axis.

1.2 Main results

Let $\mu(\beta) = \mu_{a_1,a_2}(\beta)$ denote the lowest eigenvalue of the transverse operator $-f_\beta^{-1}\frac{\mathrm{d}}{\mathrm{d}t}f_\beta\frac{\mathrm{d}}{\mathrm{d}t}$ in $L^2(\omega, f_\beta\,\mathrm{d}t)$ with Dirichlet conditions, where $f_\beta(t) := \sqrt{1+\beta^2t^2}$; that is,

$$\mu(\beta) = \inf_{0\neq\psi\in C_0^\infty(\omega)} \frac{\int_\omega f_\beta|\psi'|^2\,\mathrm{d}t}{\int_\omega f_\beta|\psi|^2\,\mathrm{d}t}. \tag{1.4}$$

In the notation of [6, (13)–(14)], the transverse eigenvalue at the section s is $\mu(\theta'(s))$.

Theorem 1.1 (Segments off the axis)**.** *Let $a_1a_2 \geq 0$ and let θ be locally Lipschitz. Put*

$$W(s,t) := \frac{1+4\theta'(s)^2t^2}{4t^2\,(1+\theta'(s)^2t^2)^2}.$$

Then for every $\psi\in\mathcal{D}(h)$,

$$h[\psi] - \Lambda_1\|\psi\|_{\mathcal{H}}^2 \;\geq\; \int_{\mathbb{R}\times\omega} W\,|\psi|^2 f\,\mathrm{d}s\,\mathrm{d}t \;\geq\; \int_{\mathbb{R}\times\omega}\frac{|\psi|^2}{4t^2f}\,\mathrm{d}s\,\mathrm{d}t. \tag{1.5}$$

Consequently $\sigma(H)\subset[\Lambda_1,\infty)$ and Λ_1 is not an eigenvalue of H. If moreover (1.3) holds, then $\inf\sigma_{\mathrm{ess}}(H) = \Lambda_1$; when in addition $a_1a_2 > 0$,

$$\sigma(H) = \sigma_{\mathrm{ess}}(H) = [\Lambda_1,\infty) \qquad \textit{and} \qquad \sigma_{\mathrm{disc}}(H) = \emptyset.$$

Theorem 1.1 answers the question of [6, Section 1.2] in the negative for every $a_1 a_2 > 0$: no strip with $a_1 a_2 > 0$ and diverging twisting has discrete eigenvalues. It proves the conjecture of [6, Section 1.2] for every $a_1 a_2 > 0$, without a largeness assumption, and for every locally Lipschitz twist. For $a_1 a_2 > 0$ the weight W decays uniformly like $\theta'(s)^{-2}$, which is the true order of the transverse spectral gap (Proposition 3.3). Hypothesis (1.3) is essential for the spectral conclusions: a twist that is large only on a long bounded interval produces discrete eigenvalues in $[\Lambda_1, E_1)$ (Remark 3.6).

For segments that meet or cross the axis, $a_1 \le 0 \le a_2$, the threshold is the disc eigenvalue $\Lambda_1 = j_{0,1}^2 / r_2^2$.

Theorem 1.2 (Segments meeting the axis)**.** *Let $a_1 \le 0 \le a_2$ and let θ be locally Lipschitz. Then for every $\psi \in \mathcal{D}(h)$,*

$$h[\psi] \;\ge\; \int_{\mathbb{R}\times\omega} \Big(\Lambda_1 + \frac{E_1 - \Lambda_1}{f}\Big) |\psi|^2 f \,\mathrm{d}s\,\mathrm{d}t. \tag{1.6}$$

In particular $\mu(\beta) > \min\{E_1, \Lambda_1\}$ for every $\beta \neq 0$. If $E_1 \ge \Lambda_1$, then $H - \Lambda_1 \ge (E_1 - \Lambda_1) f^{-1} \ge 0$, $\sigma(H) \subset [\Lambda_1, \infty)$, Λ_1 is not an eigenvalue of H, and under (1.3) $\inf \sigma_{\mathrm{ess}}(H) = \Lambda_1$.

For $a_1 a_2 < 0$ the condition $E_1 \ge \Lambda_1$ reads $a_2 - a_1 \le (\pi/j_{0,1})\, r_2$, that is, $\min\{|a_1|, a_2\} \le (\pi/j_{0,1} - 1) \max\{|a_1|, a_2\}$ with $\pi/j_{0,1} - 1 = 0.30637\ldots$ (Lemma 2.2). For $a_1 a_2 = 0$ it always holds.

Theorem 1.3 (Large twisting on asymmetric segments)**.** *Let $a_1 < 0 < a_2$ and $|a_1| \neq a_2$. There is $\beta_0 > 0$ such that $\mu(\beta) > \Lambda_1$ for all $|\beta| \ge \beta_0$; an admissible β_0 is given explicitly in Lemma 5.2. Consequently:*

(a) if $|\theta'| \ge \beta_0$ almost everywhere, then $H \ge \Lambda_1$ and Λ_1 is not an eigenvalue of H;

(b) if (1.3) *holds, then $\inf \sigma_{\mathrm{ess}}(H) = \Lambda_1$ and H has at most finitely many eigenvalues below Λ_1, counted with multiplicity.*

Theorem 1.4 (Existence criterion)**.** *Let $a_1 < a_2$.*

(i) There is a twist $\theta \in C^2(\mathbb{R})$ satisfying (1.3) *for which H has a discrete eigenvalue below Λ_1 if and only if $E_1 < \Lambda_1$, that is, if and only if $a_1 a_2 < 0$ and $a_2 - a_1 > (\pi/j_{0,1}) \max\{|a_1|, a_2\}$.*

(ii) H has a discrete eigenvalue below Λ_1 for every locally Lipschitz θ satisfying (1.3) *if and only if $a_1 = -a_2$.*

(iii) $\inf \sigma(H) < \Lambda_1$ for every locally Lipschitz θ if and only if $a_1 = -a_2$.

Corollary 1.5 (The existence theorems of [6])**.** *The conclusions of [6, Theorem 3] and [6, Theorem 2] hold for every admissible twist exactly when $a_1 = -a_2$. More precisely, among the pairs with $a_1 a_2 \le 0$:*

(a) if $a_1 = -a_2$, then $\inf \sigma(H) < \Lambda_1$ for every locally Lipschitz θ, and $\sigma_{\mathrm{disc}}(H) \cap (0, \Lambda_1) \neq \emptyset$ under (1.3)*;*

(b) if $a_1 a_2 = 0$, or if $a_1 a_2 < 0$ and $a_2 - a_1 \le (\pi/j_{0,1}) r_2$, then $H \ge \Lambda_1$ for every locally Lipschitz θ, so $\sigma(H) \cap (-\infty, \Lambda_1) = \emptyset$;

(c) if $a_1 a_2 < 0$, $a_1 \neq -a_2$ and $a_2 - a_1 > (\pi/j_{0,1}) r_2$, then $H \ge \Lambda_1$ for $\theta(s) = \beta_0 s$ and for $\theta(s) = \beta_0 s + s^3$, the latter satisfying (1.3)*, while suitable twists satisfying* (1.3) *produce discrete eigenvalues below Λ_1.*

For example, for $\omega = (-1/2, 1)$ *the helicoidal strip* $\theta(s) = 20s$ *satisfies* $H \geq \mu(20) > \Lambda_1$*, because* $20 > \beta_0 = 18.158$ *(Table 1), and* $\theta(s) = 20s + s^3$ *satisfies* (1.3) *and gives* $H \geq \Lambda_1$*. For* $\omega = (-1/5, 1)$ *and* $\theta \equiv 0$*,* $\inf \sigma(H) = E_1 \approx 6.854 > \Lambda_1 \approx 5.783$*. For* $\omega = (0, 1)$ *and* $\theta \equiv 0$*,* $\inf \sigma(H) = \pi^2 > j_{0,1}^2 = \Lambda_1$*.*

The proof of [6, Theorem 3] uses the trial function $\varphi_n(s)\psi_1(t)$, where ψ_1 is the ground state of the limiting operator [6, (19), (21), (23)]. Since $f \geq 1$, every element of $\mathcal{D}(h)$ lies in $L^2(\mathbb{R}; H_0^1(\omega))$, and $\varphi_n\psi_1 \in \mathcal{D}(h)$ exactly when $\psi_1 \in H_0^1(\omega)$, which holds exactly when $a_1 = -a_2$: for $a_1 = 0$ the function $\psi_1(t) = J_0(j_{0,1}t/a_2)$ equals 1 at the Dirichlet end $t = 0$, and for $a_1 < 0 < a_2$ with $|a_1| \neq a_2$ it vanishes on the shorter half and jumps at $t = 0$. In the symmetric case the limiting eigenvalue is double, and the argument of [6] is complete once ψ_1 is taken to be the even eigenfunction $J_0(j_{0,1}|t|/a_2)$, which is smooth (Proposition 6.1). Corollary 1.5 shows that the symmetric case is also the exact range of the statements.

On the accumulation of discrete eigenvalues in these strips [7, Open Problem 5.7], Theorems 1.1, 1.2 and 1.3(b) give the following for the spectrum below Λ_1: for every $a_1 \neq -a_2$ and every twist satisfying (1.3), it consists of finitely many eigenvalues. In every case covered by Theorem 1.1, by Theorem 1.2 with $E_1 \geq \Lambda_1$, and by Theorem 1.3(a), the threshold Λ_1 is not an eigenvalue; this bears on the third question of [6, Section 1.2], on the nature of the essential spectrum.

1.3 Relation to earlier work and route of the proofs

Twisting of a non-symmetric cross-section raises the spectrum of tubes. In three dimensions twisting that vanishes at infinity yields a Hardy inequality [3], and diverging twisting of a cross-section lying in a half-plane makes the spectrum purely discrete [5]. For strips on ruled surfaces with a symmetric cross-section and bounded twisting, negative Gauss curvature yields a Hardy inequality above E_1 [4], and higher-dimensional versions are in [7]; local twisting of thin strips creates discrete eigenvalues [1]. In the diverging regime [6] discovered the raise of dimension at infinity. For three-dimensional tubes with diverging twisting, eigenvalue asymptotics and heat traces are studied in [10, 9]; the present proofs are self-contained and use none of their results.

All our lower bounds start from the transverse estimate of [6, (12)] (Lemma 2.1), which reduces the problem to the one-dimensional quotient (1.4). Three different comparisons for $\mu(\beta)$ then settle the three geometric cases.

- Off the axis (Section 3), the substitution $v = f_\beta^{1/2}\psi$ turns the quotient (1.4) into a Schrödinger quotient with the potential of [6, (11)], and the substitution $v = t^{1/2}w$ turns the annulus quotient into one with potential $-1/(4t^2)$. The difference of the two potentials is the explicit positive function W_β of Lemma 3.1, and this gives (1.5).

- For segments meeting the axis (Section 4), the potential $-1/(4t^2)$ is not integrable at $t = 0$, and a superposition takes the place of the potential comparison: Lemma 4.1 writes $f_\beta = 1 + \int_0^\infty (|\beta t| - c)_+ \, \mathrm{d}\nu(c)$. The constant weight has Poincaré constant E_1. Each cone weight $(|t| - \rho)_+$ is the radial weight of a disc of radius at most r_2, so its constant is at least Λ_1. This gives (1.6).

- For large twisting on asymmetric crossing segments (Section 5), we write $\psi = J_0(kt)\, g$. The identity of Lemma 5.1, which extends the computation [6, (47)–(48)] to all trial functions, isolates a negative term of size $O(\beta^{-1} \log \beta)$. The Dirichlet condition at the end of the shorter half gives a capacity bound of the same size on $g(0)$, and this term is absorbed.

The existence statements (Section 6) use explicit trial functions, the lower bound $\inf\sigma_{\mathrm{ess}}(H) \geq \Lambda_1$ of [6, Lemma 1], and, in the symmetric case, the trial function of [6, Theorem 3]. Section 7 gives certified values of β_0 and a numerical illustration.

2 Preliminaries

Throughout, $\mathcal{D}(h)$ is the closure of $C_0^\infty(\mathbb{R}\times\omega)$ in the norm $(h[\psi]+\|\psi\|_{\mathcal{H}}^2)^{1/2}$. On every bounded set $(-L,L)\times\omega$ the functions f and f^{-1} are bounded, because $\theta'\in L^\infty_{\mathrm{loc}}$. Hence on such sets the form norm is equivalent to the H^1 norm, and every Lipschitz function on $\mathbb{R}\times\overline{\omega}$ with bounded support in s that vanishes at $t=a_1$ and $t=a_2$ belongs to $\mathcal{D}(h)$. Since $f\geq 1$, the form norm dominates $\big(\int(|\partial_t\psi|^2+|\psi|^2)\,\mathrm{d}s\,\mathrm{d}t\big)^{1/2}$, so $\mathcal{D}(h)\subset L^2(\mathbb{R};H_0^1(\omega))$.

2.1 The transverse problem

For each $\beta\in\mathbb{R}$ the quotient (1.4) is the bottom of the operator L_β of the closed form $\int_\omega f_\beta|\psi'|^2\,\mathrm{d}t$ on $H_0^1(\omega)$ in $L^2(\omega,f_\beta\,\mathrm{d}t)$. Since $1\leq f_\beta\leq f_{|\beta|}(r_2)$ on ω, the form domain is $H_0^1(\omega)$, L_β has compact resolvent, and $\mu(\beta)$ is a simple eigenvalue attained by a ground state. Moreover μ is even and continuous: for $|\beta-\beta'|\leq 1$ the ratio $f_\beta/f_{\beta'}$ lies in $[1-\epsilon,1+\epsilon]$ with $\epsilon\to 0$ as $\beta'\to\beta$, uniformly on ω.

Lemma 2.1 (Transverse estimate [6, (12)])**.** *For every $\psi\in\mathcal{D}(h)$,*

$$h[\psi]\;\geq\;\int_{\mathbb{R}\times\omega}\frac{|\partial_s\psi|^2}{f}\,\mathrm{d}s\,\mathrm{d}t+\int_{\mathbb{R}}\mu(\theta'(s))\int_\omega|\psi(s,t)|^2 f(s,t)\,\mathrm{d}t\,\mathrm{d}s.$$

More generally, if $\Phi(\beta,\cdot)\geq 0$ are measurable weights with $\int_\omega f_\beta|\varphi'|^2\,\mathrm{d}t\geq\int_\omega\Phi(\beta,t)f_\beta|\varphi|^2\,\mathrm{d}t$ for all $\varphi\in C_0^\infty(\omega)$ and all β, then $h[\psi]\geq\int f^{-1}|\partial_s\psi|^2+\int\Phi(\theta'(s),t)|\psi|^2 f$ on $\mathcal{D}(h)$.

Proof. For $\psi\in C_0^\infty(\mathbb{R}\times\omega)$ and almost every s, the function $\psi(s,\cdot)$ lies in $C_0^\infty(\omega)$ and $f(s,\cdot)=f_{\theta'(s)}$. Apply the transverse inequality at that s and integrate in s. For general $\psi\in\mathcal{D}(h)$ choose $\psi_n\in C_0^\infty(\mathbb{R}\times\omega)$ with $\psi_n\to\psi$ in the form norm and almost everywhere. Then $h[\psi_n]\to h[\psi]$ and $\int f^{-1}|\partial_s\psi_n|^2\to\int f^{-1}|\partial_s\psi|^2$, and Fatou's lemma bounds the last integral for ψ by the lim inf of the integrals for ψ_n. □

2.2 The threshold

Lemma 2.2 (E_1 against Λ_1)**.** *(a) If $a_1a_2\geq 0$, then $\Lambda_1<E_1$. (b) If $a_1a_2<0$, then $\Lambda_1=j_{0,1}^2/r_2^2$ and $E_1<\Lambda_1$ if and only if $a_2-a_1>(\pi/j_{0,1})r_2$; this holds when $a_1=-a_2$. The number $\pi/j_{0,1}-1$ lies in $[0.3063,0.3064]$.*

Proof. (a) This is [6, (26)]; we recall the argument. Here $E_1=\pi^2/(r_2-r_1)^2$. With $\ell:=r_2-r_1$ and $\phi(r)=\sin(\pi(r-r_1)/\ell)$ one has $\int_{r_1}^{r_2}r(\phi'^2-\pi^2\ell^{-2}\phi^2)\,\mathrm{d}r=\pi^2\ell^{-2}\int_0^\ell(r_1+x)\cos(2\pi x/\ell)\,\mathrm{d}x=0$. So the radial Rayleigh quotient of ϕ (an admissible function for the annulus, or for the disc when $r_1=0$) equals E_1, and $\Lambda_1\leq E_1$. Equality would make ϕ a radial eigenfunction, $-(r\phi')'=\Lambda_1 r\phi$; but $-\phi''=E_1\phi$, so $\phi'\equiv 0$, which is false. (b) The first statement is [6, (22)], and the equivalence is a rearrangement. For $a_1=-a_2$ one has $a_2-a_1=2r_2>(\pi/j_{0,1})r_2$ because $\pi/j_{0,1}<1.31$. The enclosure is certified in `v3_certified_constants.py`. □

Lemma 2.3 (The bottom is not an eigenvalue)**.** *Let $\Lambda\in\mathbb{R}$ and suppose that $\mu(\theta'(s))\geq\Lambda$ for almost every s. Then $H\geq\Lambda$ and Λ is not an eigenvalue of H.*

Proof. Lemma 2.1 gives $h[\psi] - \Lambda\|\psi\|^2 \geq \int f^{-1}|\partial_s\psi|^2 \geq 0$. If $H\psi = \Lambda\psi$, then $h[\psi] = \Lambda\|\psi\|^2$, hence $\partial_s\psi = 0$ almost everywhere. Since $\psi \in H^1_{\mathrm{loc}}$, $\psi(s,t) = \varphi(t)$ for some φ, and then $\|\psi\|^2 \geq \int_{\mathbb{R}}\int_\omega |\varphi|^2 \,\mathrm{d}t\,\mathrm{d}s$ is finite only if $\varphi = 0$. □

Lemma 2.4 (The threshold from above)**.** *Assume* (1.3). *Then* $\inf\sigma_{\mathrm{ess}}(H) \leq \Lambda_1$ *for every* $a_1 < a_2$.

Proof. Let $\epsilon > 0$. By the variational characterization [6, (17)] of Λ_1 over $C_0^\infty((r_1,r_2))$, and after the reflection $t \mapsto -t$ when $r_2 = |a_1|$, there is a real $w \in C_0^\infty(\omega)$ whose support lies in $\{\tau \leq |t|\}$ for some $\tau > 0$ and satisfies

$$q_w := \frac{\int_\omega |t|\,|w'|^2\,\mathrm{d}t}{\int_\omega |t|\,|w|^2\,\mathrm{d}t} < \Lambda_1 + \epsilon.$$

Fix a real $0 \neq \phi \in C_0^\infty((0,1))$. By (1.3) we may choose $s_j \to \infty$ with $s_{j+1} \geq s_j + 1$ and $|\theta'| \geq j$ almost everywhere on $[s_j, s_j+1]$. Put $\Psi_j(s,t) := \phi(s-s_j)w(t)$. On the support of Ψ_j, $x := |\theta'(s)||t| \geq j\tau$, and $x \leq f \leq x + 1/(2x)$, so $|\theta'||t| \leq f \leq (1+\eta_j)|\theta'||t|$ with $\eta_j := (2j^2\tau^2)^{-1}$. With $m_j := \int |\theta'(s)|\phi(s-s_j)^2\,\mathrm{d}s \geq j\|\phi\|^2$ this gives

$$\int f|\partial_t\Psi_j|^2 \leq (1+\eta_j)m_j\int |t||w'|^2, \quad \|\Psi_j\|^2 \geq m_j\int |t||w|^2, \quad \int \frac{|\partial_s\Psi_j|^2}{f} \leq \frac{\|\phi'\|^2}{j}\int \frac{|w|^2}{|t|}.$$

Hence $h[\Psi_j] \leq (\Lambda_1 + 2\epsilon)\|\Psi_j\|^2$ for all large j. The Ψ_j have disjoint supports, so they are orthogonal both in $\mathcal{H}$ and for h, and $h < (\Lambda_1 + 3\epsilon)\|\cdot\|^2$ on their span. By the min-max principle [8, Theorem XIII.1] the spectral subspace of H for $(-\infty, \Lambda_1 + 3\epsilon)$ is infinite-dimensional, so $\sigma_{\mathrm{ess}}(H)$ meets $(-\infty, \Lambda_1 + 3\epsilon]$. □

Lemma 2.5 (Scaling and reflection)**.** *For* $r > 0$ *and* $\beta \in \mathbb{R}$ *one has* $\mu_{a_1,a_2}(\beta) = r^{-2}\mu_{a_1/r,a_2/r}(r\beta)$ *and* $\mu_{a_1,a_2}(\beta) = \mu_{-a_2,-a_1}(\beta)$. *The operator* H *for* (a_1,a_2) *is unitarily equivalent to the operator for* $(-a_2,-a_1)$ *with the same* θ, *and* Λ_1, E_1 *are unchanged.*

Proof. Substitute $\psi(t) = \varphi(t/r)$ in (1.4), using $f_\beta(r\tau) = f_{r\beta}(\tau)$. For the reflection use that f is even in t, so $\psi(s,t) \mapsto \psi(s,-t)$ is unitary and preserves (1.2). □

3 Segments off the axis

In this section $0 \leq a_1 < a_2$; the case $a_1 < a_2 \leq 0$ follows by Lemma 2.5. For a positive function $g \in C^2(\overline{\omega})$ and $v \in C_0^\infty(\omega)$, a direct computation gives the ground-state formula

$$\int_\omega g\,|(g^{-1/2}v)'|^2\,\mathrm{d}t = \int_\omega |v'|^2\,\mathrm{d}t + \int_\omega \frac{(g^{1/2})''}{g^{1/2}}\,|v|^2\,\mathrm{d}t, \tag{3.1}$$

since the pointwise difference of the two sides is the derivative of $-\frac{1}{2}(g'/g)|v|^2$. For $g = f_\beta$ the potential is [6, (11)]

$$V_\beta(t) := \frac{(f_\beta^{1/2})''}{f_\beta^{1/2}} = \frac{\beta^2(2-\beta^2t^2)}{4(1+\beta^2t^2)^2},$$

and for $g(t) = t$ on $\omega \subset (0,\infty)$ it is $-1/(4t^2)$.

Lemma 3.1 (The potential gap)**.** *For* $t > 0$ *and* $\beta \in \mathbb{R}$,

$$W_\beta(t) := V_\beta(t) + \frac{1}{4t^2} = \frac{1+4\beta^2t^2}{4t^2(1+\beta^2t^2)^2}, \qquad W_\beta(t) - \frac{1}{4t^2(1+\beta^2t^2)} = \frac{3\beta^2}{4(1+\beta^2t^2)^2} \geq 0,$$

and $W_\beta(t) \leq \beta^{-2}t^{-4}$ *for* $\beta \neq 0$.

Proof. With $x = \beta^2 t^2$ one has $x(2-x) + (1+x)^2 = 1 + 4x$, which gives the first identity, and $(1+4x) - (1+x) = 3x$ gives the second. Finally $(1+4x)/(1+x)^2 \le 4/(1+x) < 4/x$. The identities are also checked symbolically in `v1_identities.py`. □

Lemma 3.2 (Transverse identity)**.** *Let $0 \le a_1 < a_2$, $\beta \in \mathbb{R}$, $\psi \in C_0^\infty(\omega)$, and $w := (f_\beta/t)^{1/2}\psi \in C_0^\infty(\omega)$. Then*

$$\int_\omega f_\beta |\psi'|^2 \,\mathrm{d}t - \Lambda_1 \int_\omega f_\beta |\psi|^2 \,\mathrm{d}t = \Big(\int_\omega t|w'|^2 \,\mathrm{d}t - \Lambda_1 \int_\omega t|w|^2 \,\mathrm{d}t \Big) + \int_\omega W_\beta f_\beta |\psi|^2 \,\mathrm{d}t.$$

Proof. Put $v := f_\beta^{1/2}\psi = t^{1/2} w$; it has compact support in ω, which lies in $(0,\infty)$ (for $a_1 = 0$ the support of ψ stays away from 0). Apply (3.1) with $g = f_\beta$ and with $g = t$:

$$\int f_\beta |\psi'|^2 = \int |v'|^2 + \int V_\beta |v|^2, \qquad \int t|w'|^2 = \int |v'|^2 - \int \frac{|v|^2}{4t^2}.$$

Subtract, and use $\int f_\beta|\psi|^2 = \int |v|^2 = \int t|w|^2$ and $|v|^2 = f_\beta |\psi|^2$. □

Proposition 3.3 (The transverse gap)**.** *Let $0 \le a_1 < a_2$ and $\beta \in \mathbb{R}$. Then*

$$\mu(\beta) - \Lambda_1 \;\ge\; \inf_{t\in\omega} W_\beta(t) \;\ge\; \frac{1}{4a_2^2(1+\beta^2 a_2^2)}.$$

If $a_1 > 0$ and $\beta \ne 0$, then also $\mu(\beta) - \Lambda_1 \le \int_\omega W_\beta \, t\phi_A^2 \,\mathrm{d}t / \int_\omega t\phi_A^2 \,\mathrm{d}t \le C_A \beta^{-2}$, where $\phi_A > 0$ is the radial ground state of the annulus and $C_A := \int_\omega t^{-3}\phi_A^2 \,\mathrm{d}t / \int_\omega t\phi_A^2 \,\mathrm{d}t \le a_1^{-4}$.

Proof. By [6, (17)], $\int t|w'|^2 \ge \Lambda_1 \int t|w|^2$ for $w \in C_0^\infty(\omega)$, so Lemma 3.2 gives $\int f_\beta|\psi'|^2 \ge (\Lambda_1 + \inf W_\beta)\int f_\beta|\psi|^2$. By Lemma 3.1, $W_\beta(t) \ge 1/(4t^2(1+\beta^2 t^2))$, which is decreasing in t. For the upper bound, ϕ_A is smooth on $\overline{\omega}$ and vanishes at a_1 and a_2. The function $\psi_A := (t/f_\beta)^{1/2}\phi_A$ lies in $H_0^1(\omega)$, and the computation of Lemma 3.2 applies to it, because $v = t^{1/2}\phi_A$ vanishes at both ends and the boundary terms in (3.1) vanish. The bracket vanishes for $w = \phi_A$. Then use $W_\beta \le \beta^{-2}t^{-4}$ from Lemma 3.1. □

Proof of Theorem 1.1. Lemma 3.2 and [6, (17)] give $\int f_\beta|\varphi'|^2 \ge \int (\Lambda_1 + W_\beta) f_\beta |\varphi|^2$ for $\varphi \in C_0^\infty(\omega)$, and Lemma 2.1 with $\Phi(\beta,t) = \Lambda_1 + W_\beta(t)$ yields the first inequality in (1.5). The second follows from Lemma 3.1. Since $W \ge 0$, $H \ge \Lambda_1$, so $\sigma(H) \subset [\Lambda_1, \infty)$. By Proposition 3.3, $\mu \ge \Lambda_1$, so Lemma 2.3 shows that Λ_1 is not an eigenvalue. Under (1.3), Lemma 2.4 gives $\inf \sigma_{\mathrm{ess}}(H) \le \Lambda_1 \le \inf \sigma(H)$, hence equality. If $a_1 a_2 > 0$, [6, Theorem 1] gives $\sigma_{\mathrm{ess}}(H) = [\Lambda_1, \infty)$, and then $\sigma(H) \subset [\Lambda_1,\infty) = \sigma_{\mathrm{ess}}(H)$ forces $\sigma(H) = \sigma_{\mathrm{ess}}(H)$ and $\sigma_{\mathrm{disc}}(H) = \emptyset$. □

Remark 3.4. For $a_1 a_2 > 0$ the eigenfunctions ϕ_1^m of the radial operators $-|t|^{-1}\partial_t |t| \partial_t + m^2 t^{-2}$ used in the proof of [6, Lemma 2] are smooth on $\overline{\omega}$ and satisfy the Dirichlet conditions, so the singular sequences there belong to $\mathcal{D}(h)$. This is the part of [6, Theorem 1] that the last step uses.

Remark 3.5. Proposition 3.3 shows $\mu(\beta) - \Lambda_1 \asymp \beta^{-2}$ as $|\beta| \to \infty$ when $a_1 a_2 > 0$. On an interval of length ℓ where $\theta' \equiv \beta$, the trial functions $\chi(s)\psi_\beta(t)$, with $\chi(s)$ a sine arch on the interval and ψ_β the transverse ground state, satisfy $h - \Lambda_1 \|\cdot\|^2 \le (\mu(\beta) - \Lambda_1 + \pi^2\ell^{-2})\|\cdot\|^2$. Hence every nonnegative weight ϱ with $H - \Lambda_1 \ge \varrho$ has a weighted mean over such an interval of at most $C_A\beta^{-2} + \pi^2\ell^{-2}$, and the decay $\theta'(s)^{-2}$ of W is the optimal order for twists that are constant on long intervals. For $a_1 = 0$ the upper bound of Proposition 3.3 is not available, W behaves like $1/(4t^2)$ near $t = 0$, and the transverse gap is much larger; for $\omega = (0,1)$ the computed value of $\mu(1000) - \Lambda_1$ is 1.13 (script `v2_transverse_numerics.py`).

Remark 3.6 (Diverging twisting is essential). Let $a_1 a_2 \geq 0$, so that $\Lambda_1 < E_1$ by Lemma 2.2. For every $w \in C_0^\infty(\omega)$ supported away from $t = 0$ one has $f_\beta/(|\beta||t|) \to 1$ uniformly on the support of w as $|\beta| \to \infty$, so [6, (17)] gives $\limsup_{|\beta|\to\infty} \mu(\beta) \leq \Lambda_1$. Fix β with $\mu(\beta) < E_1$ and $L > 0$, and let $\theta(s) := \beta \max\{-L, \min\{s, L\}\}$, a Lipschitz twist with $\theta' = \beta \mathbf{1}_{(-L,L)}$. Since $\theta' \to 0$ at infinity, $\sigma_{\mathrm{ess}}(H) = [E_1, \infty)$ by [6, (25)]. The trial function $\cos(\pi s/(2L)) \mathbf{1}_{\{|s|<L\}} \psi_\beta(t)$, with ψ_β the transverse ground state, gives $\inf \sigma(H) \leq \mu(\beta) + \pi^2/(4L^2)$, because $\int f_\beta^{-1} \psi_\beta^2 \leq \int f_\beta \psi_\beta^2$. For large L this is less than E_1, so H has a discrete eigenvalue, which lies in $[\Lambda_1, E_1)$ by Theorem 1.1.

4 Segments meeting the axis

In this section $a_1 \leq 0 \leq a_2$, so $r_2 = \max\{|a_1|, a_2\}$ and $\Lambda_1 = j_{0,1}^2/r_2^2$ by [6, (22)]. Let ν be the probability measure $\mathrm{d}\nu(c) = (1 + c^2)^{-3/2}\,\mathrm{d}c$ on $(0, \infty)$; its total mass and its first moment both equal 1.

Lemma 4.1 (Superposition of cone weights). *For every $x \geq 0$, $\sqrt{1 + x^2} = 1 + \int_0^\infty (x - c)_+ \,\mathrm{d}\nu(c)$. Consequently $f_\beta(t) = 1 + \int_0^\infty (|\beta t| - c)_+ \,\mathrm{d}\nu(c)$ for all $\beta, t \in \mathbb{R}$.*

Proof. $\int_0^x (x - c)(1 + c^2)^{-3/2}\,\mathrm{d}c = x \cdot \frac{x}{\sqrt{1+x^2}} - \left(1 - \frac{1}{\sqrt{1+x^2}}\right) = \sqrt{1 + x^2} - 1.$ □

Lemma 4.2 (Cone weights). *Let $\rho \geq 0$ and $\psi \in H_0^1(\omega)$. Then*

$$\int_\omega (|t| - \rho)_+ |\psi'|^2 \,\mathrm{d}t \;\geq\; \Lambda_1 \int_\omega (|t| - \rho)_+ |\psi|^2 \,\mathrm{d}t.$$

Moreover, if $0 \leq \rho < |a_1| < a_2$, then the left side minus the right side is at least $\left(j_{0,1}^2/(|a_1| - \rho)^2 - \Lambda_1\right) \int_{a_1}^{-\rho} (|t| - \rho)|\psi|^2 \,\mathrm{d}t$.

Proof. By density it suffices to take $\psi \in C_0^\infty(\omega)$. If $a_2 > \rho$, put $R := a_2 - \rho$ and $\phi(r) := \psi(\rho + r)$ for $0 \leq r \leq R$. Then ϕ is smooth on $[0, R]$ and vanishes near R, so $x \mapsto \phi(|x|)$ is a Lipschitz function with compact support in the disc of radius R. The Dirichlet eigenvalue of that disc is $j_{0,1}^2/R^2$, which gives

$$\int_\rho^{a_2} (t - \rho)|\psi'|^2 \,\mathrm{d}t = \int_0^R r|\phi'|^2 \,\mathrm{d}r \;\geq\; \frac{j_{0,1}^2}{R^2} \int_0^R r|\phi|^2 \,\mathrm{d}r \;\geq\; \Lambda_1 \int_\rho^{a_2} (t - \rho)|\psi|^2 \,\mathrm{d}t,$$

since $R \leq r_2$. The half $(a_1, -\rho)$ is treated in the same way, with $R = |a_1| - \rho$. If a half is empty its contribution is zero. Adding the two halves gives the claim, and the refinement records the larger constant on the shorter half. □

Proposition 4.3 (Interpolation inequality). *For $\beta \in \mathbb{R}$ and $\psi \in H_0^1(\omega)$,*

$$\int_\omega f_\beta |\psi'|^2 \,\mathrm{d}t \;\geq\; E_1 \int_\omega |\psi|^2 \,\mathrm{d}t + \Lambda_1 \int_\omega (f_\beta - 1)|\psi|^2 \,\mathrm{d}t. \tag{4.1}$$

Consequently $\mu(\beta) \geq \min\{E_1, \Lambda_1\}$, with strict inequality for $\beta \neq 0$.

Proof. By Lemma 4.1 and Tonelli's theorem,

$$\int f_\beta |\psi'|^2 = \int |\psi'|^2 + \int_0^\infty |\beta| \int_\omega \big(|t| - c/|\beta|\big)_+ |\psi'|^2 \,\mathrm{d}t \,\mathrm{d}\nu(c)$$

for $\beta \neq 0$, and the same with ψ in place of ψ'. Apply the Poincaré inequality $\int |\psi'|^2 \geq E_1 \int |\psi|^2$ to the first term and Lemma 4.2 with $\rho = c/|\beta|$ to the inner integrals. This proves (4.1); for $\beta = 0$ it is the Poincaré inequality. Since the right side of (4.1) is at least $m := \min\{E_1, \Lambda_1\}$ times $\int f_\beta |\psi|^2$, we get $\mu(\beta) \geq m$.

Let $\beta \neq 0$ and suppose $\mu(\beta) = m$, attained by a ground state $\psi \neq 0$. Both inequalities used above hold separately and $f_\beta > 1$ for $t \neq 0$, so $\int |\psi|^2 > 0$ and $\int (f_\beta - 1)|\psi|^2 > 0$. Equality therefore forces $E_1 = \Lambda_1 = m$, equality in the Poincaré inequality, and equality in the cone inequalities for ν-almost every c. The first forces $\psi = C \sin(\pi(t - a_1)/(a_2 - a_1))$ with $C \neq 0$. By Lemma 2.2, $E_1 = \Lambda_1$ requires $a_1 < 0 < a_2$ and $|a_1| \neq a_2$ (for $a_1 a_2 = 0$, $E_1 > \Lambda_1$; for $a_1 = -a_2$, $E_1 < \Lambda_1$). By Lemma 2.5 we may assume $|a_1| < a_2$. Then the refinement in Lemma 4.2 gives a strict cone inequality for the positive function ψ whenever $c/|\beta| < |a_1|$, a set of positive ν-measure. This is a contradiction. □

Proof of Theorem 1.2. The right side of (4.1) equals $\int_\omega \Phi(\beta, t) f_\beta |\psi|^2 \, \mathrm{d}t$ with $\Phi(\beta, t) := \Lambda_1 + (E_1 - \Lambda_1)/f_\beta(t) = (1 - f_\beta^{-1})\Lambda_1 + f_\beta^{-1} E_1$, a convex combination of Λ_1 and E_1, so $\Phi \geq \min\{E_1, \Lambda_1\} > 0$. Lemma 2.1 with this weight gives (1.6). The strict bound on μ is Proposition 4.3. If $E_1 \geq \Lambda_1$, then (1.6) reads $H - \Lambda_1 \geq (E_1 - \Lambda_1) f^{-1} \geq 0$, so $\sigma(H) \subset [\Lambda_1, \infty)$. Moreover $\mu \geq \Lambda_1$ everywhere, so Lemma 2.3 applies, and under (1.3) Lemma 2.4 gives $\inf \sigma_{\mathrm{ess}}(H) = \Lambda_1$. □

Remark 4.4. For $a_1 = 0$ both Theorem 1.1 and Theorem 1.2 apply. They give two Hardy inequalities above $\Lambda_1 = j_{0,1}^2/a_2^2$: one with the weight W, which behaves like $1/(4t^2)$ near the edge $t = 0$, and one with the weight $(\pi^2 - j_{0,1}^2) a_2^{-2} f^{-1}$.

5 Large twisting on asymmetric crossing segments

In this section $a_1 < 0 < a_2$ and $|a_1| \leq a_2$; the case $|a_1| > a_2$ follows by Lemma 2.5. Put $k := j_{0,1}/a_2$, so that $\Lambda_1 = k^2$, and

$$u(t) := J_0(kt), \qquad q(t) := \frac{k\, J_1(k|t|)}{|t|} \quad (t \neq 0), \qquad q(0) := \frac{k^2}{2}.$$

We use the following standard facts [2, §§10.6, 10.14, 10.21]: $J_0' = -J_1$; J_0 is positive and decreasing on $[0, j_{0,1})$; $J_1 > 0$ on $(0, j_{1,1})$ with $j_{0,1} < j_{1,1}$; $|J_0| \leq 1$; and $|J_1(x)| \leq |x|/2$. Hence u is even, smooth and positive on $(-a_2, a_2)$, with $u(a_2) = 0$, and $0 < q \leq k^2/2$ on $\overline{\omega}$.

Lemma 5.1 (Bessel ground-state identity)**.** *For $\beta \in \mathbb{R}$, $\psi \in C_0^\infty(\omega)$ and $g := \psi/u \in C_0^\infty(\omega)$,*

$$Q_\beta[\psi] := \int_\omega f_\beta |\psi'|^2 \, \mathrm{d}t - \Lambda_1 \int_\omega f_\beta |\psi|^2 \, \mathrm{d}t = \int_\omega f_\beta u^2 |g'|^2 \, \mathrm{d}t - \int_\omega \frac{q}{f_\beta}\, u\, |g|^2 \, \mathrm{d}t.$$

If $a_1 = -a_2$, the same identity holds for $\psi = u$ and $g \equiv 1$, that is, $Q_\beta[u] = -\int_\omega (q/f_\beta) u \, \mathrm{d}t < 0$.

Proof. For $t \neq 0$ the function u satisfies Bessel's equation $u'' + u'/t + k^2 u = 0$, and $f_\beta' - f_\beta/t = (\beta^2 t^2 - f_\beta^2)/(t f_\beta) = -1/(t f_\beta)$. Hence

$$-(f_\beta u')' - k^2 f_\beta u = -u' \Big(f_\beta' - \frac{f_\beta}{t} \Big) = \frac{u'}{t f_\beta} = -\frac{q}{f_\beta},$$

because $u'(t)/t = -k J_1(kt)/t = -q(t)$, and the identity extends to $t = 0$ by continuity. For real g, expanding $\int f_\beta |(ug)'|^2$ and integrating the cross term $\int f_\beta u u' (g^2)'$ by parts gives $\int f_\beta |(ug)'|^2 =$

$\int f_\beta u^2 g'^2 - \int (f_\beta u')' u g^2$. The boundary terms vanish because g has compact support, or, for $\psi = u$ in the symmetric case, because $u(\pm a_2) = 0$. Subtracting $k^2 \int f_\beta u^2 g^2$ gives

$$Q_\beta[ug] = \int f_\beta u^2 g'^2 + \int \left(-(f_\beta u')' - k^2 f_\beta u\right) u\, g^2 = \int f_\beta u^2 g'^2 - \int \frac{q}{f_\beta}\, u\, g^2.$$

Complex g is handled by taking real and imaginary parts. The integrated identity is also checked by 30-digit quadrature in `v1_identities.py`. □

For $a_1 = -a_2$ the second statement is the computation behind [6, (47)–(50)]. Indeed, with the function $W = 1/(tf^2)$ of [6, (48)], the quantity $(\varphi\psi_1, W\varphi\psi_1')_{\mathcal{H}}$ equals $-\int \varphi^2 \int (q/f)\psi_1 \,\mathrm{d}t\,\mathrm{d}s$. For $|a_1| < a_2$ the negative term must be controlled by the positive one. We normalize $a_2 = 1$ by Lemma 2.5, write $a := |a_1| \in (0,1)$, and fix $\delta \in (0,1)$. Put

$$A(\beta) := \frac{\operatorname{arsinh}(\beta a)}{\beta}, \quad B(\beta) := \frac{\operatorname{arsinh}(\beta(1-\delta))}{\beta}, \quad u_a := J_0(j_{0,1}a), \quad u_\delta := J_0(j_{0,1}(1-\delta)),$$

and let m_δ and M_δ be the minimum and maximum of J_1 on $[j_{0,1}(1-\delta), j_{0,1}]$; these are positive.

Lemma 5.2 (The large-twist bound)**.** *Let $\omega = (-a, 1)$ with $0 < a < 1$, let $\beta > 0$, and put*

$$\kappa(\beta) := \frac{j_{0,1}^2 A^2}{2u_a^2} + j_{0,1}^2 B\Big(\frac{A}{u_a^2} + \frac{B}{u_\delta^2}\Big) + \frac{j_{0,1}^3 M_\delta}{2\beta(1-\delta)}\Big[2\delta^2\Big(\frac{A}{u_a^2} + \frac{B}{u_\delta^2}\Big) + \frac{\delta}{j_{0,1}^2 m_\delta^2 \beta(1-\delta)}\Big].$$

Then $\int_\omega (q/f_\beta) u|g|^2 \le \kappa(\beta) \int_\omega f_\beta u^2 |g'|^2$ for all $g \in C_0^\infty(\omega)$. The function κ is decreasing on $(0,\infty)$ and tends to 0. If $\kappa(\beta_0) < 1$, then $\mu(\beta) > \Lambda_1$ for all $|\beta| \ge \beta_0$.

Proof. Write $E := \int_\omega f_\beta u^2 |g'|^2$, E_S for the part of this integral over $S := (-a, 0)$, and $c := g(0)$. We use $\int_0^a f_\beta^{-1} = A$, $\int_0^{1-\delta} f_\beta^{-1} = B$, $u \le 1$ and $q \le j_{0,1}^2/2$.

The shorter half. On S we have $u \ge u_a$, and g vanishes near $-a$. By Cauchy–Schwarz,

$$\sup_S |g|^2 \le \int_{-a}^0 \frac{\mathrm{d}t}{f_\beta u^2}\; E_S \le \frac{A}{u_a^2} E_S;$$

in particular $|c|^2 \le (A/u_a^2)E$. Hence $\int_S (q/f_\beta) u|g|^2 \le \frac{j_{0,1}^2}{2} \cdot \frac{A}{u_a^2} E \cdot A$.

The core of the longer half. On $B_0 := (0, 1-\delta)$ we have $u \ge u_\delta$, so $|g(t)|^2 \le 2|c|^2 + 2(B/u_\delta^2)E$. Hence $\int_{B_0} (q/f_\beta) u|g|^2 \le j_{0,1}^2 B\,(|c|^2 + BE/u_\delta^2)$.

The end of the longer half. On $B_1 := (1-\delta, 1)$ put $x := 1 - t$ and $G(x) := g(1-x)$; G vanishes near $x = 0$. Since $u(t) = \int_{j_{0,1}t}^{j_{0,1}} J_1$, we have $j_{0,1} m_\delta x \le u \le j_{0,1} M_\delta x$ there, and $f_\beta \ge \beta(1-\delta)$. Hence $(q/f_\beta)u \le j_{0,1}^3 M_\delta x/(2\beta(1-\delta))$ on B_1. From $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{2}x^2|G|^2) = x|G|^2 + x^2 \operatorname{Re}(\overline{G}G')$, the Cauchy–Schwarz inequality and $\int_0^\delta x^2|G|^2 \le \delta \int_0^\delta x|G|^2$, absorbing gives

$$\int_0^\delta x|G|^2 \,\mathrm{d}x \le \delta^2 |G(\delta)|^2 + \delta \int_0^\delta x^2 |G'|^2 \,\mathrm{d}x.$$

Here $\int_0^\delta x^2|G'|^2 = \int_{B_1} (1-t)^2 |g'|^2 \le E/(j_{0,1}^2 m_\delta^2 \beta(1-\delta))$ and $|G(\delta)|^2 = |g(1-\delta)|^2 \le 2|c|^2 + 2BE/u_\delta^2$. Therefore

$$\int_{B_1} \frac{q}{f_\beta} u|g|^2 \le \frac{j_{0,1}^3 M_\delta}{2\beta(1-\delta)} \int_0^\delta x|G|^2 \,\mathrm{d}x \le \frac{j_{0,1}^3 M_\delta}{2\beta(1-\delta)}\Big[2\delta^2\Big(|c|^2 + \frac{BE}{u_\delta^2}\Big) + \frac{\delta E}{j_{0,1}^2 m_\delta^2 \beta(1-\delta)}\Big].$$

Adding the three bounds and inserting $|c|^2 \le (A/u_a^2)E$ gives the inequality with κ. Each term of κ is a product of positive decreasing functions, because $\operatorname{arsinh}(y)/y$ is decreasing, and $A, B = O(\beta^{-1}\log\beta)$, so $\kappa \to 0$.

If $\kappa(\beta) < 1$, then by Lemma 5.1 $Q_\beta[\psi] \ge (1-\kappa(\beta))E \ge 0$ for $\psi \in C_0^\infty(\omega)$, so $\mu(\beta) \ge \Lambda_1$. Suppose $\mu(\beta) = \Lambda_1$, and take $\psi_n \in C_0^\infty(\omega)$ with $\int f_\beta|\psi_n|^2 = 1$ and $Q_\beta[\psi_n] \to 0$. Then $E_n \to 0$. By the bounds above, $g_n \to 0$ uniformly on $[-a, 1-\delta]$, and $\int_{B_1} u^2|g_n|^2 \le j_{0,1}^2 M_\delta^2 \delta \int_0^\delta x|G_n|^2 \to 0$. Hence $\int f_\beta|\psi_n|^2 \to 0$, which is a contradiction. Thus $\mu(\beta) > \Lambda_1$ whenever $\kappa(\beta) < 1$, and in particular for $\beta \ge \beta_0$ by monotonicity. Negative β follow because μ is even. □

Lemma 5.3 (Neumann bracketing)**.** *Let $s_0 > 0$. Let h_{int} and h_{ext} be the closures of the expression* (1.2) *on the restrictions of $C_0^\infty(\mathbb{R}\times\omega)$ to $(-s_0, s_0)\times\omega$ and to $(\mathbb{R}\setminus[-s_0,s_0])\times\omega$, with operators H_{int} and H_{ext}. Then for every $\lambda \in \mathbb{R}$,*

$$N(H,\lambda) \le N(H_{\mathrm{int}},\lambda) + N(H_{\mathrm{ext}},\lambda),$$

where $N(T,\lambda)$ is the dimension of the spectral subspace of T for $(-\infty,\lambda)$.

Proof. The restriction map $J\psi := (\psi|_{\mathrm{int}}, \psi|_{\mathrm{ext}})$ is an isometry of $\mathcal{H}$ into the orthogonal sum of the two Hilbert spaces. It maps $\mathcal{D}(h)$ into $\mathcal{D}(h_{\mathrm{int}} \oplus h_{\mathrm{ext}})$ and satisfies $(h_{\mathrm{int}} \oplus h_{\mathrm{ext}})[J\psi] = h[\psi]$. By the min-max principle [8, Theorem XIII.1], $N(T,\lambda)$ is the largest dimension of a subspace of the form domain on which the form is less than $\lambda\|\cdot\|^2$, and J maps such subspaces injectively; see also [8, Section XIII.15]. □

Proof of Theorem 1.3. By Lemma 2.5 we may assume $a_2 = 1 > |a_1| = a$. Lemma 5.2 gives β_0; for general (a_1, a_2) the admissible value is $\beta_0(a_1/r_2, a_2/r_2)/r_2$. (a) If $|\theta'| \ge \beta_0$ almost everywhere, then $\mu(\theta'(s)) > \Lambda_1$ for almost every s, and Lemma 2.3 applies. (b) By (1.3) choose s_0 with $|\theta'| \ge \beta_0$ almost everywhere on $|s| > s_0$. Lemma 2.1 holds verbatim for h_{ext}, since its proof is fibrewise in s, so $H_{\mathrm{ext}} \ge \Lambda_1$ and $N(H_{\mathrm{ext}}, \Lambda_1) = 0$. On the bounded rectangle $(-s_0, s_0)\times\omega$ the form h_{int} is equivalent to the H^1 form with Dirichlet conditions at $t = a_1, a_2$. Since H^1 of a rectangle embeds compactly into L^2, H_{int} has compact resolvent and $N(H_{\mathrm{int}}, \Lambda_1) < \infty$. Lemma 5.3 gives $N(H, \Lambda_1) < \infty$, so the spectrum below Λ_1 consists of finitely many eigenvalues of finite multiplicity, and $\inf\sigma_{\mathrm{ess}}(H) \ge \Lambda_1$. Lemma 2.4 gives equality. □

6 The existence criterion

Proposition 6.1 (The symmetric case [6, Theorem 3])**.** *Let $a_1 = -a_2$. Then $\inf\sigma(H) < \Lambda_1$ for every locally Lipschitz θ. Under* (1.3)*, H has a discrete eigenvalue in $(0, \Lambda_1)$.*

Proof. This is the argument of [6, Section 4], which is complete in this case once ψ_1 is taken to be the even eigenfunction $J_0(j_{0,1}|t|/a_2)$ of the (doubly degenerate) limiting eigenvalue. Let $\varphi_n(s) := \min\{1, (2 - |s|/n)_+\}$ and $\Psi_n(s,t) := \varphi_n(s)u(t)$. Here $u(t) = J_0(j_{0,1}|t|/a_2)$ is smooth on $\overline{\omega}$ and vanishes at $\pm a_2$, so $\Psi_n \in \mathcal{D}(h)$. By Lemma 5.1 applied for each s, with $f \ge 1$,

$$h[\Psi_n] - \Lambda_1\|\Psi_n\|^2 = \int \frac{|\varphi_n'|^2 u^2}{f} - \int_{\mathbb{R}} \varphi_n(s)^2 \int_\omega \frac{q}{f} u \,\mathrm{d}t\,\mathrm{d}s \le \frac{2}{n}\int_\omega u^2\,\mathrm{d}t - c_*,$$

where $c_* := \int_{-1}^1 \int_\omega (q/f)u\,\mathrm{d}t\,\mathrm{d}s > 0$ for $n \ge 1$, since $u > 0$ in ω and f is bounded on $[-1,1]\times\omega$. This is negative for large n. Under (1.3), $\inf\sigma_{\mathrm{ess}}(H) \ge \Lambda_1$ by [6, Lemma 1]. So $\inf\sigma(H)$ is a discrete eigenvalue below Λ_1, and it is positive because $H \ge \min\{E_1, \Lambda_1\} > 0$ by Theorem 1.2. □

Proof of Theorem 1.4. (i) Suppose $E_1 < \Lambda_1$. By Lemma 2.2, $a_1 a_2 < 0$ and $a_2 - a_1 > (\pi/j_{0,1})r_2$. For $L > 0$ let $\theta_L(s) := ((|s| - L)_+)^3$. It is C^2, satisfies (1.3), and $\theta_L' = 0$ on $[-L, L]$, where $f = 1$. The trial function

$$\Psi(s,t) := \cos\Big(\frac{\pi s}{2L}\Big)\mathbf{1}_{\{|s|<L\}} \sin\Big(\frac{\pi(t-a_1)}{a_2-a_1}\Big) \in \mathcal{D}(h)$$

gives $h[\Psi] = (E_1 + \pi^2/(4L^2))\|\Psi\|^2$. For $L > \pi/(2\sqrt{\Lambda_1 - E_1})$ this is less than $\Lambda_1\|\Psi\|^2$, so $\inf\sigma(H) < \Lambda_1 \le \inf\sigma_{\rm ess}(H)$ by [6, Lemma 1]. (For $a_1 \neq -a_2$ the last inequality is also Theorem 1.3(b).) Hence $\inf\sigma(H)$ is a discrete eigenvalue below Λ_1. Conversely, suppose $E_1 \ge \Lambda_1$. If $a_1 a_2 \ge 0$, Theorem 1.1 gives $H \ge \Lambda_1$ for every θ. If $a_1 a_2 < 0$, Theorem 1.2 gives the same. In both cases no twist produces spectrum below Λ_1.

(ii) and (iii). If $a_1 = -a_2$, Proposition 6.1 gives both properties. Let $a_1 \neq -a_2$. If $E_1 \ge \Lambda_1$, then $H \ge \Lambda_1$ for every θ, as in (i). If $E_1 < \Lambda_1$, then $a_1 a_2 < 0$ and $|a_1| \neq a_2$, and Theorem 1.3(a) gives $H \ge \Lambda_1$ both for $\theta(s) = \beta_0 s$ and for $\theta(s) = \beta_0 s + s^3$, which satisfies (1.3). For these twists the properties in (iii) and in (ii), respectively, fail. □

Proof of Corollary 1.5. Part (a) is Proposition 6.1. Part (b) is Theorem 1.1 for $a_1 a_2 = 0$ and Theorem 1.2 for $a_1 a_2 < 0$, using Lemma 2.2. Part (c) is the proof of Theorem 1.4. For $\omega = (0,1)$ and $\theta \equiv 0$, H is the Dirichlet Laplacian of the flat strip $\mathbb{R}\times(0,1)$, whose spectrum is $[\pi^2, \infty)$, while $\Lambda_1 = j_{0,1}^2$. □

Example 6.2. For $a_1 a_2 < 0$ the flat strip $\theta \equiv 0$ has $\inf\sigma(H) = E_1$. Hence the conclusion of [6, Theorem 3] fails at $\theta \equiv 0$ exactly when $\min\{|a_1|, a_2\} \le (\pi/j_{0,1} - 1)\max\{|a_1|, a_2\}$; for instance $\omega = (-1/5, 1)$ gives $E_1 = \pi^2/1.44 \approx 6.854 > \Lambda_1 \approx 5.783$. When the ratio exceeds $\pi/j_{0,1} - 1$ and $a_1 \neq -a_2$, the twist $\theta(s) = \beta_0 s$ of Theorem 1.3 gives the same conclusion.

7 Certified constants and a numerical illustration

Table 1 lists, for $\omega = (-a, 1)$ and $\delta = 1/2$, values β_0 with $\kappa(\beta_0) < 1$ certified in outward-rounded interval arithmetic at the printed (rounded-up) value. The Bessel functions are enclosed by their power series with explicit tail bounds, and $m_{1/2} \ge 0.4972$, $M_{1/2} \le 0.5858$ (script `v3_certified_constants.py`). By Lemma 5.2 these values are admissible in Theorem 1.3. For comparison, the table gives the largest grid value of β at which a finite-difference computation of $\mu(\beta) - \Lambda_1$ is negative (script `v2_transverse_numerics.py`; floating point, an illustration only). For $a \le \pi/j_{0,1} - 1$ no such β exists, by Theorem 1.2.

a	0.31	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
certified β_0	13.901	15.501	18.158	22.683	31.408	51.803	124.678	293.285	1939.361
last negative grid β	0.25	1.58	2.51	3.55	4.47	6.31	10.0	14.1	35.5

Table 1: Admissible β_0 in Theorem 1.3 for $\omega = (-a, 1)$, and the observed sign change of $\mu(\beta) - \Lambda_1$.

The same script checks, for $\omega \in \{(1,2), (0.05,1), (2,2.1), (0.3,3)\}$, the two-sided bound of Proposition 3.3 in the discretized problem, where it holds exactly by Weyl's inequalities. It also checks $\mu(\beta) > \min\{E_1, \Lambda_1\}$ for $\beta \in [0.01, 1000]$ on several segments meeting the axis, and $\mu(\beta) < \Lambda_1$ on $(-1, 1)$. A finite-element computation of the best constant in Lemma 5.2 (script `v4_kappa_check.py`) gives, for $a = 1/2$, the values 0.62, 0.12 and 0.011 at $\beta = 5, 20, 100$, against $\kappa = 4.9$, 0.87 and 0.077; by Lemma 5.1 the best constant crosses 1 exactly where $\mu(\beta) - \Lambda_1$ changes sign, and the computed crossing lies between $\beta = 2.5$ and $\beta = 3$. The pointwise identities of

Lemmas 3.1, 4.1 and 5.1 and the formula (3.1) are verified symbolically, and the integrated identity of Lemma 5.1 by 30-digit quadrature, in `v1_identities.py`.

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