Bose gases in weakly twisted annular-sector waveguides

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Abstract

We study an ideal Bose gas in an infinite-volume Dirichlet tube whose annular-sector cross-section rotates with speed $c(1+z^2)^{p/2}$, as $c \downarrow 0$. At particle scale $c^{-1/p}$ the critical scaled density is finite exactly when $0 < p < 1$; above it the excess occupies the single ground mode. The critical density retains every transverse band of an explicit operator-valued symbol. For $p > 1$, a denser particle scale $c^{-2/(p+1)}$ instead yields positive limiting occupation fractions in every fixed mode of an anharmonic oscillator. The marginal logarithmic scale is also identified, using a uniform two-parameter harmonic comparison specific to the sector. We prove the actual fixed-mode spectral limit, an intermediate-energy count and the full fixed-time heat-trace limit, and control the singular excited Bose occupation as the spectral gap closes. All statistical statements use grand-canonical mean occupations. The power-trap criteria and free-Gibbs occupation laws are classical; the result supplies their precise geometric realization for this nonseparable confining family.

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1 Setting and conclusions

The opening limit weakens the twisting coefficient c while retaining unbounded twisting at spatial infinity for every fixed $c > 0$. Each operator therefore has compact resolvent on an infinite-volume tube. The particle normalizations describe that opening family, rather than ordinary particle number per volume.

The three regimes are read with their scales attached. At the natural $c^{-1/p}$ particle scale, $0 < p < 1$ has a finite critical scaled density. For $p > 1$, the denser $c^{-2/(p+1)}$ scale produces the fixed-mode oscillator law. At $p = 1$, the sector-specific harmonic comparison yields the logarithmically denser $c^{-1}\log(1/c)$ normalization. These statements concern grand-canonical mean occupations; a normalized occupation becomes a fraction of the total only after division by the prescribed scaled particle number.

Fix

$$0 < r_0 < r_1 < \infty, \qquad 0 < a < 2\pi, \qquad p > 0. \tag{1.1}$$

Put $\beta_c(z) = c(1+z^2)^{p/2}$ and $\theta_c(z) = \int_0^z \beta_c(s)\,\mathrm{d}s$. Our domain is

$$\Omega_c = \{(r\cos(t+\theta_c(z)), r\sin(t+\theta_c(z)), z) : r_0 < r < r_1,\ 0 < t < a,\ z \in \mathbb{R}\}.$$

The parametrization is injective because its third coordinate fixes z and the angular width is less than 2π. The tube is connected and has infinite volume. Let H_c be its Friedrichs Dirichlet

Laplacian. With $D = (r_0, r_1) \times (0, a)$ and transverse measure $r\,\mathrm{d}r\,\mathrm{d}t$, the unitary pullback has form

$$q_c[u] = \int_{\mathbb{R}} q_D[u(\cdot, z)]\,\mathrm{d}z + \|\partial_z u - \beta_c(z)\partial_t u\|^2\,, \qquad q_D[v] = \int_D (|v_r|^2 + r^{-2}|v_t|^2) r\,\mathrm{d}r\,\mathrm{d}t. \quad (1.2)$$

The form domain is the closure of $C_c^\infty(D \times \mathbb{R})$ in its form norm. The transverse operator H_D has simple ground energy e_0 and a real normalized positive ground state ϕ. Write

$$\kappa = \|\partial_t \phi\|^2 = (\pi/a)^2, \quad h = c^{1/p}, \quad \delta = c^{2/(p+1)}, \quad \nu = \frac{p+1}{2p}, \qquad h = \delta^\nu.$$

The value of κ follows by separation: the lowest angular mode is $\sin(\pi t/a)$, since higher modes increase the radial centrifugal potential. The low-mode and symbol arguments only require $\kappa > 0$; the marginal comparison in Section 6 additionally uses $\phi_{tt} = -\kappa\phi$.

For $k \in \mathbb{R}$, $b \geq 0$, define the transverse operator $H(k, b)$ by its fixed-domain form

$$q_{k,b}[v] = q_D[v] + \|ikv - b\partial_t v\|^2\,, \qquad v \in H_0^1(D). \quad (1.3)$$

For inverse temperature $\beta > 0$ let

$$F_\beta(s) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{Tr}_D \left(e^{\beta(H(k,|y|^p) - e_0 + s)} - 1 \right)^{-1} \mathrm{d}k\,\mathrm{d}y, \qquad s > 0, \quad (1.4)$$

$$\rho_c(\beta) = \lim_{s \downarrow 0} F_\beta(s). \quad (1.5)$$

The subscript c in ρ_c means critical and does not denote the opening parameter. The quantity depends on p, D, β.

Theorem 1.1. *The operator H_c has compact resolvent for every $c > 0$ and simple ground level $\lambda_1(c)$. List its eigenvalues with multiplicity. For each $\rho > 0$ there is a unique $\mu_c < \lambda_1(c)$ such that*

$$h \sum_{j \geq 1} n_j(c) = \rho, \qquad n_j(c) = \frac{1}{e^{\beta(\lambda_j(c) - \mu_c)} - 1}. \quad (1.6)$$

The critical scaled density is finite exactly when $0 < p < 1$. If $\rho < \rho_c(\beta)$, including every finite ρ when $p \geq 1$, then $\mu_c \to e_0 - s_\rho$, where $F_\beta(s_\rho) = \rho$, and $hn_1(c) \to 0$. If $0 < p < 1$ and $\rho \geq \rho_c(\beta)$, then

$$\mu_c \to e_0, \qquad h \sum_{j \geq 2} n_j(c) \to \rho_c(\beta), \qquad hn_1(c) \to \rho - \rho_c(\beta).$$

For $\rho > \rho_c(\beta)$,

$$\lambda_1(c) - \mu_c \sim \frac{h}{\beta(\rho - \rho_c(\beta))}. \quad (1.7)$$

For $0 < p < 1$, the normalized excited occupation in a shrinking window above the actual ground energy vanishes uniformly as specified in (5.1)*. Every fixed excited mode has $hn_j(c) \to 0$ for all $p > 0$: for $p \geq 1$ this follows from the fixed positive limiting chemical gap.*

There is another natural particle normalization. Let E_j be the increasing eigenvalues of

$$A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \kappa|x|^{2p} \quad \text{on } L^2(\mathbb{R}). \quad (1.8)$$

Theorem 1.2. *Suppose $p > 1$. Choose μ_c by $\delta \sum_j n_j(c) = \rho > 0$. There is a unique $r_\rho > 0$ satisfying*

$$G_\beta(r_\rho) = \rho, \qquad G_\beta(r) = \frac{1}{\beta} \sum_{j \geq 1} (E_j - E_1 + r)^{-1}.$$

Then

$$\frac{\lambda_1(c)-\mu_c}{\delta}\longrightarrow r_\rho,\qquad \delta n_j(c)\longrightarrow\frac{1}{\beta(E_j-E_1+r_\rho)}$$

for every fixed j, and the limiting occupations sum to ρ. Thus infinitely many fixed modes have positive limiting fractions at this denser scale, with no positive critical density at this scale.

Theorems 1.1 and 1.2 concern an ideal noninteracting gas and grand-canonical mean occupations. At $p=1$, Theorem 6.1 treats the denser logarithmic particle normalization $c^{-1}\log(1/c)$, including its critical coefficient and chemical-potential exponent.

Ancestry and scope

Geometric confinement by twisting and the characteristic half-tail estimate used below are established for these polar strips in [1]; its high-energy Weyl exponent is different from the infrared exponent ν here. Constant-twist Fourier fibers are classical [3]. The angular-momentum Born–Huang twisting potential is explicit in [4, Section 4.4.2]; the effective potential in (1.8) is an instance of it. Power-trap condensation criteria go back to [5, 6]. Geometry-induced Bose condensation and the spectral-gap occupation argument already appear in [2]. Rigorous harmonic open-trap thermodynamics is developed in [7]. The scalar free-Gibbs high-temperature inverse-operator limit [8, Lemma 3.4] is the statistical ancestor of Theorem 1.2. The harmonic logarithmic normalization is treated in [9].

The contribution here is the complete opening-limit analysis of the actual globally confining, nonseparable twisted tube: its full-band thermal symbol, closing low-level gaps and uniform singular-occupation control. A scalar effective power potential alone does not determine (1.4) at fixed temperature.

2 Characteristic confinement and the low spectrum

Lemma 2.1 (Whole half-tails)**.** *If $\beta_c\ge B>0$ on $z>Z$, then the form with free trace at Z and lateral Dirichlet trace satisfies*

$$\|(\partial_z-\beta_c\partial_t)u\|^2\ge\frac{\pi^2B^2}{4a^2}\,\|u\|^2\,.$$

The negative half-tail has the same bound. For a function supported outside $[-Z,Z]$ and vanishing at the cuts, the constant improves from $\pi^2/(4a^2)$ to π^2/a^2.

Proof. Set $\vartheta=t+\int^z\beta_c(s)\,\mathrm{d}s$. At fixed r,ϑ the covariant derivative becomes the vertical derivative; the Jacobian is one. A permitted connected vertical passage has length at most a/B. Its angular endpoints are Dirichlet. Only a passage meeting the flat cut can have a free endpoint, and a half-tail has at most one such endpoint per passage. Apply the one-dimensional Dirichlet–Neumann Poincare inequality, then integrate $r\,\mathrm{d}r\,\mathrm{d}\vartheta$. With zero trace at the cut both endpoints are Dirichlet. The argument first applies to smooth vectors with the indicated traces and passes to the form domain by approximation. Equivalently one may use the unwrapped angle; no angular periodic boundary condition is imposed. □

Local boundedness of β_c makes (1.2) equivalent to an ordinary H^1 norm on bounded cylinders. Lemma 2.1 and $\beta_c(z)\to\infty$ give uniform tail tightness for form-bounded sets, so Rellich proves compact resolvent. Positivity on the connected Dirichlet domain gives simplicity of its ground state.

Proposition 2.2. *For every fixed j, $(\lambda_j(c)-e_0)/\delta\to E_j$. In particular $\lambda_1=e_0+\delta E_1+o(\delta)$ and $\lambda_2-\lambda_1=\delta(E_2-E_1)+o(\delta)$, with $E_2>E_1$.*

Proof. Write $\varepsilon = c^{1/(p+1)}$, so $\delta = \varepsilon^2$, and set $z = x/\varepsilon$. The exact scaled excess form is

$$a_\varepsilon[v] = \varepsilon^{-2} \int_{\mathbb{R}} (q_D[v] - e_0 \|v\|_D^2)\, \mathrm{d}x + \|D_\varepsilon v\|^2\,, \quad D_\varepsilon = \partial_x - b_\varepsilon(x)\partial_t, \quad b_\varepsilon = (\varepsilon^2 + x^2)^{p/2}. \tag{2.1}$$

Both terms are nonnegative. The Dirichlet characteristic bound for exterior-supported vectors gives

$$\|D_\varepsilon v\|^2 \geq (\pi/a)^2 R^{2p} \|v\|^2 \quad \text{if } \operatorname{supp} v \subset \{|x| > R\}.$$

For a cutoff $\chi_R = 0$ on $|x| \leq R$, $\chi_R = 1$ on $|x| \geq 2R$, $|\chi_R'| \leq C/R$, the product rule yields

$$\int_{|x|\geq 2R} |v|^2 \leq C R^{-2p} (\|D_\varepsilon v\|^2 + R^{-2} \|v\|^2). \tag{2.2}$$

Suppose $a_{\varepsilon_i}[v_i] + \|v_i\|^2$ is bounded and $\varepsilon_i \to 0$. If P is the fiberwise projection onto ϕ and $e_1 > e_0$ is the next transverse eigenvalue, then

$$\|(1-P)v_i\|^2 \leq \frac{\varepsilon_i^2 a_{\varepsilon_i}[v_i]}{e_1 - e_0} \to 0.$$

The transverse energy is bounded, and on every bounded x-interval $\partial_x v_i = D_{\varepsilon_i} v_i + b_{\varepsilon_i} \partial_t v_i$ is bounded in L^2. Local Rellich and (2.2) give a subsequence converging strongly in global L^2 to ϕf. The covariant derivatives have a weak L^2 limit, identified locally in distributions as

$$W = \phi f' - |x|^p \phi_t f.$$

No derivative of b_ε is needed. Since $\langle \phi, \phi_t \rangle_D = 0$, projection of W onto ϕ proves $f \in H^1(\mathbb{R})$; the orthogonal component proves $|x|^p f \in L^2$. Weak lower semicontinuity gives

$$\|f'\|^2 + \kappa \||x|^p f\|^2 \leq \liminf_i a_{\varepsilon_i}[v_i].$$

For recovery take $v = \phi f$ with $f \in C_c^\infty(\mathbb{R})$:

$$a_\varepsilon[\phi f] = \|f'\|^2 + \kappa \int (\varepsilon^2 + x^2)^p |f|^2 \,\mathrm{d}x \longrightarrow \|f'\|^2 + \kappa \||x|^p f\|^2\,.$$

Approximation of ϕ in $H_0^1(D)$ justifies this trial in the closed domain. Cutoff and mollification give recovery for the full effective form domain. The oscillator has compact form embedding; its eigenvalues are simple by one-dimensional Sturm–Liouville theory. Recovery on finite-dimensional trial spaces proves the eigenvalue limsup. For the liminf take the first j normalized actual eigenvectors on a liminf subsequence. The compactness just proved gives j orthonormal limits $\phi f_1, \ldots, \phi f_j$. Apply the liminf inequality to each fixed linear combination and use minmax on their span. This proves the matching bound and the result. □

3 Fibers and a uniform intermediate count

Lemma 3.1. *The lowest fiber eigenvalue has its unique global minimum e_0 at $(k, b) = (0, 0)$, and locally*

$$E(k, b) - e_0 = k^2 + \kappa b^2 + O((k^2 + b^2)^2). \tag{3.1}$$

For every $\tau > 0$ the phase heat integral is finite. The critical quantity in (1.4) *is finite exactly for $p < 1$.*

Proof. Angular differentiation is form-bounded by q_D because $r \le r_1$. The fixed-domain forms are analytic in k, b near zero. Their perturbation is homogeneous of degree two; its first expectation is $k^2 + \kappa b^2$, since $\langle \phi, \partial_t \phi \rangle = 0$. Analytic perturbation of the simple eigenvalue gives (3.1). All higher bands have a positive gap in a neighborhood of zero. The inequality $H(k,b) \ge H_D$ gives a lower bound e_0. Equality would require the transverse vector to be ϕ and $ik\phi - b\phi_t = 0$, whose norm squared is $k^2 + \kappa b^2$.

For global control reserve half the transverse energy, complete the angular square at each r, and use angular Dirichlet Poincare:

$$H(k,b) \ge \tfrac{1}{2} H_D + \frac{\pi^2 b^2}{a^2} + \frac{k^2}{1 + 2b^2 r_1^2}. \tag{3.2}$$

Indeed the angular coefficient is $b^2 + (2r^2)^{-1}$; its gauge Poincare lower bound includes $\pi^2 b^2 / a^2$, while the residual scalar is $k^2/(1 + 2b^2 r^2)$. The gauge may depend on r but is used only in this angular expression, not in a transformed radial energy. Minmax implies

$$\operatorname{Tr}_D e^{-\tau H(k,b)} \le \operatorname{Tr}_D e^{-\tau H_D/2} \exp\left(-\tau \left[\frac{\pi^2 b^2}{a^2} + \frac{k^2}{1 + 2b^2 r_1^2} \right] \right). \tag{3.3}$$

After k integration this is bounded by a constant times $(1 + |y|^p) e^{-C|y|^{2p}}$, with $b = |y|^p$. The low-energy parameter sets are compact by (3.2); strict positivity gives a positive gap off each neighborhood of zero. There the Bose factor is bounded by a heat factor. Higher bands near zero are controlled likewise. The first band is comparable to

$$\int_{|k|,|y|<\eta} \frac{\mathrm{d}k \, \mathrm{d}y}{k^2 + |y|^{2p}},$$

which is finite exactly when $\int_0^\eta y^{-p} \, \mathrm{d}y$ is finite. This proves the assertion, including logarithmic divergence at $p = 1$. □

Proposition 3.2. *There exist $C, E_*, c_* > 0$ such that*

$$N(H_c, e_0 + E) \le C h^{-1} E^\nu, \qquad 0 < c < c_*, \quad \delta \le E \le E_*. \tag{3.4}$$

Here N counts eigenvalues strictly below its argument.

Proof. Choose a fixed large L and set $R = L(\sqrt{E}/c)^{1/p}$. Since $E \ge \delta$, $R \ge L c^{-1/(p+1)}$. Make only two natural Neumann cuts, at $\pm R$; the enlarged three-piece form domain gives an upper counting bound. On the whole tails, Lemma 2.1 and $q_D - e_0 \ge 0$ give excess energy at least $\pi^2 L^{2p} E / (4a^2) > E$ when L is large. On the central interval $R \ge 1$ for small c, and $\beta_c^2 \le 2^p L^{2p} E$. Using $|v - w|^2 \ge |v|^2/2 - |w|^2$ and $\|u_t\|^2 \le r_1^2 q_D[u]$, for a fixed $\gamma > 0$ we obtain

$$q_c - e_0 \|u\|^2 \ge \tfrac{1}{2} \|u_z\|^2 + (1 - \gamma E)(q_D - e_0 \|u\|^2) - \gamma e_0 E \|u\|^2 .$$

Shrink E_* so $1 - \gamma E \ge 1/2$ and the next transverse band of this tensor comparison lies above E. Only the simple transverse ground band can contribute. Its longitudinal Neumann count is at most $1 + CR\sqrt{E}$. Finally $R\sqrt{E} = L h^{-1} E^\nu$ and $h^{-1} \delta^\nu = 1$, which absorbs the constant. All comparisons use free trace only at the two flat cuts; no half-tail passage has two free endpoints. □

4 The full fixed-time heat trace

Proposition 4.1. *For every fixed $\tau > 0$,*

$$h \operatorname{Tr} e^{-\tau(H_c - e_0)} \longrightarrow Z(\tau) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \operatorname{Tr}_D e^{-\tau(H(k,|y|^p) - e_0)} \, \mathrm{d}k \, \mathrm{d}y. \tag{4.1}$$

These traces are finite and uniformly bounded after normalization.

Proof. Rescale $y = hz$; the form becomes $q_D + \|hu_y - b_h(y)u_t\|^2$, $b_h = (h^2 + y^2)^{p/2}$. For $|b| \le B$, angular-square minimization gives

$$q_D + \|hu_y - bu_t\|^2 \ge \tfrac{1}{2} q_D + (1 + 2r_1^2 B^2)^{-1} h^2 \|u_y\|^2 . \tag{4.2}$$

This holds for zero, periodic or free endpoint traces in y. On bounded intervals the domains are the corresponding ordinary H^1 domains with transverse zero trace and compact embeddings. Also $N_D(E) \le C(1 + E)$ by comparison with a Dirichlet rectangle.

For $E \ge 1$ choose $R_E = LE^{1/(2p)}$, L fixed large, and cut at $\pm R_E$ with natural Neumann conditions. Lemma 2.1 in the rescaled variables gives whole-tail bottoms at least $c_0 R_E^{2p} > E$, $c_0 = \pi^2/(4a^2)$. Centrally $b_h^2 \le C(1 + E)$, so separated counting using (4.2) gives

$$hN(H_c, E) \le C(1 + E)^M \qquad (0 < h \le 1, E \ge 0) \tag{4.3}$$

for some finite M: the central count is at most $C(1 + E)[1 + CR_E\sqrt{E(1 + E)}/h]$, and monotonicity treats $E < 1$. The same bound, with constants independent of $R \ge 1$, holds for each exterior Neumann operator on $y > R$ or $y < -R$: if $R < R_E$, make one additional cut at R_E and count the finite strip; if $R \ge R_E$ there are no levels below E. Its bottom is at least $c_0 R^{2p}$, hence integration of (4.3) gives

$$h \operatorname{Tr} e^{-\tau H^+_{h,R}} + h \operatorname{Tr} e^{-\tau H^-_{h,R}} \le C_\tau e^{-\tau c_0 R^{2p}/2}. \tag{4.4}$$

Polynomial counting also controls high-energy heat tails uniformly.

Fix a constant b on an unscaled longitudinal interval of length T. Denote its Dirichlet, periodic and natural Neumann realizations by $A_D(T, b), A_P(T, b), A_N(T, b)$; domains are nested in that order. Fourier series give exactly

$$\operatorname{Tr} e^{-\tau A_P(T,b)} = \sum_{\ell \in \mathbb{Z}} \operatorname{Tr}_D e^{-\tau H(2\pi\ell/T, b)}.$$

Fiber traces are continuous by fixed-domain form comparison, minmax and the summable bound $H(k, b) \ge H_D$. The Gaussian bound in k from (3.2), uniformly on compact b sets, proves

$$T^{-1} \operatorname{Tr} e^{-\tau A_P(T,b)} \longrightarrow \Phi_\tau(b) := \frac{1}{2\pi} \int_{\mathbb{R}} \operatorname{Tr}_D e^{-\tau H(k,b)} \, \mathrm{d}k. \tag{4.5}$$

We verify that the endpoint realizations have the same limit. For $T > 8s$, $s \ge 1$ fixed, choose an IMS partition on the Neumann interval consisting of a Dirichlet bulk and two endpoint collars of length $2s$, with derivative-square sum at most $C/s^2 =: \eta_s$. On the periodic circle take a Dirichlet bulk and a single collar across the join, unfolded to an interval of length $4s$. The covariant IMS error is exactly $\sum \|\chi_i' u\|^2$. The isometric localization map and minmax yield

$$\begin{aligned} N_N(E; T, b) &\le N_D(E + \eta_s; T, b) + 2N_N(E + \eta_s; 2s, b), \\ N_P(E; T, b) &\le N_D(E + \eta_s; T, b) + N_N(E + \eta_s; 4s, b). \end{aligned}$$

Integration against $\tau e^{-\tau E} \, \mathrm{d}E$ bounds the heat traces by the analogous sums multiplied by $e^{\tau\eta_s}$. Collar traces are finite and independent of T, uniformly on compact b sets by (4.2). Domain nesting, (4.5), first $T \to \infty$ and then $s \to \infty$ show that both endpoint traces per length tend to $\Phi_\tau(b)$. On a y-box of fixed length l, $T = l/h$, so

$$h \operatorname{Tr} e^{-\tau A_{h, D/N}(l,b)} \longrightarrow l\Phi_\tau(b). \tag{4.6}$$

If $|b_h - b| \le \omega$ on a box, the square inequality with parameter ω and $\|u_t\|^2 \le r_1^2 q_D$ give

$$(1 - C\omega) q_{h,b} \le q_{h,b_h} \le (1 + C\omega) q_{h,b}$$

on either endpoint domain, for small ω. Partition $[-R, R]$ into finitely many fixed y-intervals, freeze $b = |y_j|^p$, and use $b_h \to |y|^p$ uniformly there. Dirichlet partitioning gives lower heat bounds and Neumann partitioning upper bounds. Formula (4.6) at times $\tau(1 \pm C\omega)$ gives Riemann sums. Joint continuity of $\Phi_\tau(b)$ on positive compact time intervals follows from Gaussian domination. First $h \to 0$, then mesh $\to 0$, yields

$$h \operatorname{Tr} e^{-\tau H_{h,[-R,R],D/N}} \longrightarrow \int_{-R}^{R} \Phi_\tau(|y|^p)\, \mathrm{d}y.$$

Dirichlet restriction gives the global lower bound; two Neumann cuts give the upper bound plus (4.4). Let $R \to \infty$ and use (3.3) for the phase tail. Multiplication by $e^{\tau e_0}$ gives (4.1). □

5 Singular occupations and the two scales

5.1 Proof of Theorem 1.1

Write $t_j = \lambda_j - \lambda_1$, $j \geq 2$, and $M_c(t) = \#\{j \geq 2 : t_j \leq t\}$. Propositions 2.2 and 3.2 give $M_c(t) = 0$ below $b\delta$ for a fixed $b > 0$, and $M_c(t) \leq Ch^{-1}t^\nu$ above that gap, for small t. Indeed apply (3.4) at an energy slightly greater than $\max(\delta, t + \lambda_1 - e_0)$; constants absorb fixed multiples of δ. This also handles strict versus nonstrict counting. For $p < 1$, $\nu > 1$, Stieltjes integration gives

$$h \sum_{0<t_j\leq\eta} \frac{1}{e^{\beta t_j} - 1} \leq \frac{h}{\beta} \left(\frac{M_c(\eta)}{\eta} + \int_0^\eta \frac{M_c(t)}{t^2}\, \mathrm{d}t \right) \leq C_\beta \eta^{\nu-1}. \tag{5.1}$$

The actual gap is needed at zero. This also bounds the excited occupation at any admissible μ_c.

At a fixed positive chemical gap s, expand the Bose factor into its exponential series. The normalized trace at $n\beta$ is bounded by that at $\beta/2$, since $H_c \geq e_0$; $e^{-n\beta s}$ permits dominated convergence using Proposition 4.1. The limit is $F_\beta(s)$. Centering at λ_1 instead of e_0 does not change a fixed-gap limit because $\lambda_1 - e_0 \to 0$.

To pass to zero gap define the excited measure

$$m_c^{\mathrm{ex}} = h \sum_{j\geq 2} \delta_{t_j}.$$

Its positive-time Laplace moments tend to $Z(\tau)$: the full measure differs by h and acquires only the factor $e^{\tau(\lambda_1 - e_0)}$. For continuous f compactly supported in $(0, \infty)$, write $f(t) = e^{-qt} g(e^{-t})$, $q > 0$, extending g continuously to $[0, 1]$. Polynomial approximation uses the moments $q + n$; its error is bounded by a uniform heat moment. Thus the measures converge against such f to the fiber-band pushforward of $(2\pi)^{-1}\, \mathrm{d}k\, \mathrm{d}y$, whose Laplace transform is Z by Tonelli. Approximate energy cutoffs at continuity points. The high-energy Bose tail above R is bounded by $Ce^{-\beta R/2} \int e^{-\beta t/2}\, \mathrm{d}m_c^{\mathrm{ex}}$; (5.1) controls the low-energy tail. Therefore, when $p < 1$,

$$h \sum_{j\geq 2} (e^{\beta t_j} - 1)^{-1} \longrightarrow \rho_c(\beta). \tag{5.2}$$

The ground atom was removed before applying the singular zero-gap Bose function; its mass tending to zero alone would not justify this.

For fixed c, the occupation sum is continuous strictly increasing in $\mu < \lambda_1$, vanishes at $-\infty$ and diverges at λ_1 through the simple ground term. Heat finiteness proves existence and uniqueness in (1.6). The function F_β decreases continuously from $\rho_c(\beta)$ to zero. If $\rho < \rho_c$, fixed-gap convergence squeezes μ_c to $e_0 - s_\rho$, so its normalized ground term vanishes. This

proves all finite-density statements for $p \geq 1$. If $p < 1$ and $\rho \geq \rho_c$, fixed-gap comparison forces $\mu_c \to e_0$. The excited sum is bounded above by (5.2), and below by its value at every fixed positive gap for all small c. Let that gap decrease to zero. Its limit is ρ_c; subtraction from the fixed total gives the ground limit, including zero at equality. A positive excess and the small-argument expansion of the ground Bose factor give (1.7). Finally $h/\delta = c^{(1-p)/(p(p+1))} \to 0$ for $p < 1$; the excitation gap bounds each fixed excited occupation, and (5.1) excludes hidden mass.

5.2 Proof of Theorem 1.2

Here $\nu < 1$. Put $s_j(c) = (\lambda_j - \lambda_1)/\delta$, $s_j = E_j - E_1$. For fixed $r > 0$, Proposition 2.2 gives convergence of every finite partial sum in

$$\delta \sum_j \frac{1}{e^{\beta\delta(s_j(c)+r)} - 1} \longrightarrow G_\beta(r). \tag{5.3}$$

Fix $\eta < E_*/2$. The count and $\lambda_1 - e_0 = O(\delta)$ imply $\#\{j : s_j(c) \leq S\} \leq C(1+S)^\nu$ while $\delta S \leq \eta$, for small c. Stieltjes integration with $\delta/(e^{\beta\delta(s+r)} - 1) \leq 1/[\beta(s+r)]$ bounds the contribution above $S \geq 1$ and below this cutoff by $CS^{\nu-1}$, uniformly; its upper endpoint term has the same bound. For original excess at least η, the Bose factor is at most $C_{\eta,\beta} e^{-\beta(\lambda_j - e_0)/2}$. Multiplication by δ, Proposition 4.1 and $\delta/h \to 0$ make that contribution vanish. This proves (5.3) and convergence of the limiting series. The count passes to the limit; integration gives $G_\beta(r) \leq C(r^{-1} + r^{\nu-1})$, hence a zero limit at infinity. The ground term diverges at $r \downarrow 0$. Continuity and strict decrease follow by domination on compact positive r-intervals. Squeeze the actual chemical gap between fixed r values, then take each fixed mode. The tail estimate shows that their limiting occupations sum to ρ.

6 The marginal logarithmic scale

Specialize to $p = 1$ and set

$$\zeta_c = \frac{c}{\log(1/c)}, \qquad A_0 = \frac{1}{2\sqrt{\kappa}}, \qquad \rho_* = \frac{A_0}{\beta} = \frac{a}{2\pi\beta}.$$

Theorem 6.1. *At $p = 1$ the nonstrict count satisfies*

$$\frac{cN_c(E)}{E} \longrightarrow A_0 \quad \textit{jointly as } E \to 0,\ c/E \to 0, \qquad N_c(E) = \#\{j : \lambda_j - e_0 \leq E\}. \tag{6.1}$$

Moreover

$$\zeta_c \sum_{j \geq 2} (e^{\beta(\lambda_j - \lambda_1)} - 1)^{-1} \longrightarrow \rho_*.$$

Choose μ_c by $\zeta_c \sum_j n_j(c) = \rho > 0$. If $0 < \rho < \rho_*$, then $\mu_c < e_0$ eventually,

$$\frac{\log(e_0 - \mu_c)}{\log c} \longrightarrow \rho/\rho_*, \qquad \zeta_c n_1(c) \longrightarrow 0.$$

If $\rho \geq \rho_*$, the normalized excited occupation tends to ρ_* and $\zeta_c n_1(c) \to \rho - \rho_*$. For $\rho > \rho_*$,

$$\lambda_1 - \mu_c \sim \frac{c}{\beta(\rho - \rho_*)\log(1/c)}.$$

Proof. The sector identity $\phi_{tt} = -\kappa\phi$ provides more than fixed-mode convergence. On a finite longitudinal interval with Dirichlet endpoint traces write $u = \phi f + v$, with $\langle \phi, v(\cdot, z)\rangle_D = 0$. The transverse form has no mixed term. Integration by parts gives, with conjugations according to the inner-product convention,

$$\langle D_c(\phi f), D_c v\rangle = \int (\beta_c' f + 2\beta_c f')\, \langle \phi_t, v\rangle_D \, \mathrm{d}z, \qquad D_c = \partial_z - \beta_c \partial_t.$$

The β_c^2 term vanishes exactly because $\langle \phi_t, v_t\rangle = -\langle \phi_{tt}, v\rangle = \kappa \langle \phi, v\rangle = 0$. The endpoint terms vanish by Dirichlet traces. Since $|\beta_c'| \le c$, the transverse gap $\gamma > 0$ absorbs the mixed term by Cauchy–Schwarz, yielding

$$\begin{aligned} q_c[u] - e_0 \|u\|^2 \ge (1 - C\beta_{\max}^2) \|f'\|^2 + \kappa \|\beta_c f\|^2 - Cc^2 \|f\|^2 \\ + \tfrac{\gamma}{2} \|v\|^2 + \|D_c v\|^2 . \end{aligned} \tag{6.2}$$

Here no separate estimate of v_z is assumed.

For energy E use an IMS partition with central support $|z| < 2R$ and tail supports $z > R, z < -R$, where $R = B\sqrt{E}/c$ and B is fixed large. The error is $d_{c,E} \le Cc^2/E$. All localized functions have Dirichlet cutoff traces. The two-Dirichlet characteristic bound gives tail excess at least $\kappa B^2 E$; choose it larger than $4E$. For small c/E the tails do not contribute below $E + d_{c,E}$. Centrally $\beta_{\max}^2 \le c^2 + 4B^2E$. Put $b_{c,E} = 1 - C(c^2 + 4B^2E) > 0$. Drop the positive constant part of $\kappa\beta_c^2 = \kappa c^2 + \kappa c^2 z^2$ in (6.2). Below $\gamma/2$ comparison with

$$\left(-b_{c,E} \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \kappa c^2 z^2 - Cc^2 \right) \oplus (\gamma/2) I$$

bounds the count. This is valid even though the second summand is infinite dimensional: it has no spectral subspace below the threshold. Extending the scalar Dirichlet interval form to the whole line increases its count. The exact harmonic eigenvalues give

$$N_c(E) \le \frac{E + Cc^2/E + Cc^2}{2c\sqrt{b_{c,E}\kappa}} + O(1). \tag{6.3}$$

The exact trials $u = \phi f$ using whole-line harmonic eigenfunctions give conversely

$$\lambda_j - e_0 \le (2j-1)c\sqrt{\kappa} + \kappa c^2, \qquad N_c(E) \ge E/(2c\sqrt{\kappa}) - O(1) - O(c).$$

As $b_{c,E} \to 1$ and $(c^2/E)/E = (c/E)^2 \to 0$, this proves (6.1). It also gives $N_c(Kc) \le C_K$ for each sufficiently large fixed K.

Let $M_c(s)$ be the ground-centered excited count. Since $\lambda_1 - e_0 = O(c)$, removing one level and shifting the energy change $cM_c(s)/s$ by $O(1/K)$ on $s \ge Kc$. Thus for every $\epsilon > 0$ choose K large and η small so that, for all small c,

$$(A_0 - \epsilon)s/c \le M_c(s) \le (A_0 + \epsilon)s/c, \qquad Kc \le s \le \eta.$$

The first excitation gap is at least dc. Below Kc there are at most C_K modes and their occupation is $O_K(c^{-1})$; above η the uniform heat bound gives $O_{\eta,\beta}(c^{-1})$. On $(Kc, \eta]$, with $f_\beta(s) = (e^{\beta s} - 1)^{-1}$, Stieltjes integration has endpoint terms $O(c^{-1})$ and integral $\int_{Kc}^{\eta} [-f_\beta'(s)] M_c(s)\, \mathrm{d}s$. Since $-s^2 f_\beta'(s) \to 1/\beta$, its leading coefficient is $(A_0/(\beta c)) \log(\eta/(Kc))$. Multiply by ζ_c, first take $c \to 0$ at fixed K, η, then decrease the tolerance and η. This proves the maximum excited limit.

For each fixed $b \in (0,1)$ the trial $\mu = e_0 - c^b$ has normalized total occupation tending to $b\rho_*$. The low endpoint range is bounded by C_K/c^b, the high range by $O_{\eta,\beta}(c^{-1})$; both are

negligible after normalization. The middle Stieltjes integral, up to an arbitrarily small relative error, is

$$\frac{A_0}{\beta c}\int_{Kc}^{\eta}\frac{E}{(E+c^b)^2}\,\mathrm{d}E=\frac{bA_0}{\beta c}\log(1/c)+O_{K,\eta}(c^{-1});$$

its boundary terms are $O(c^{-1})$ or smaller. Monotonicity with exponents bracketing ρ/ρ_* proves the subcritical logarithmic exponent and eventual $\mu_c < e_0$. A bound $e_0-\mu_c \geq c^{b'}$, $b'<1$, makes the normalized ground term at most $\zeta_c/(\beta c^{b'})\to 0$. For $\rho\geq\rho_*$ every trial exponent $b<1$ is eventually below the actual chemical potential. The excited occupation therefore has lower limit at least $b\rho_*$ and upper limit ρ_* by its maximum. Let $b\uparrow 1$, subtract from the fixed total, and expand the positive ground Bose factor above criticality. □

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Data and materials availability. The results are analytic. The manuscript, complete proof source and accompanying reproducibility materials are available in the source package in the Luminary Research Archive.

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