Beyond logarithmic powers: critical-tail localization of theta correction inertia

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Luminary Oddities

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Abstract

For the Riemann theta transform $F(z) = \xi((1+iz)/2)/4$, let $E(P) = (4\pi)^{-1}\int |(FP)'|^2$, $Q(P) = (8\pi)^{-1}\int (F'^2 - FF'')P^2$ and $D = E - Q$. We obtain ordinary-polynomial subspaces, in either parity, with opposite signs of D and strictly positive original Q on their entire nonzero spans. Their dimensions are at least $C\exp(c\sqrt{\log N})\sqrt{\log N}$ for every sufficiently large degree N. A critical Meixner–Pollaczek receiver controls the derivative and absolute-potential norm globally. A simultaneous inverse-square window selection and a relative root-product budget improve the cost of localized targets. A finite-order compact-cutoff family with uniform low-order energies and a fully controlled growing differentiation order gives the stated scale. All local complex roots, multiplicities, exterior roots and mixed Gram entries are retained. The same growth is necessary for repair rank in a specified scalar-centered strict-contraction scheme.

1 Introduction and statements

The correction in a theta polynomial-square form can be indefinite while the original form remains positive. The physical-form predecessor [1] isolates this distinction and establishes finite-height positivity and selected-zero correction inertia. The subsequent paper [2] proves, for every fixed $A > 0$, whole-span opposite correction channels of dimension at least $c_A(\log N)^A(\log\log N)^2$. Each power uses its own fixed approximation order. That result does not justify choosing an unspecified order as a function of N.

Here a critical-tail receiver replaces that order-dependent approximation step. The classical MP generating function gives a strip-to-circle transform; its boundary phases permit polynomial approximation of localized targets after their critical exponential factor is removed. Because the theta multiplier has polynomial growth after this removal, we pay spatial moments and prove convergence in the full original norm. The local geometry selects seeds with both bounded root-product loss and bounded outer inverse-square crowding. A relative inner budget pays only the extra resolution beyond the microscopic starting scale.

The ingredients have substantial ancestry. Kuznetsov, Inoue and Romik [9, 10, 11] already expand xi or completed L-functions in MP polynomials. Hardy-strip orthogonal bases are classical [8]. The varying finite-order cutoff follows the classical Ehrenpreis–Hörmander principle [17, Section 3]. Fixed nearanalytic Fourier bumps have the Güntürk–DeVore construction recorded in [12, Appendix B, Proposition 13] and the classical Ingham framework [13]; our final rate uses a finite-order family instead. The contribution is the quantitative map to arbitrary localized F-times-polynomial targets in the original global joint norm, with the complete theta root and ordinary-degree resources. Existing xi expansions and generic weighted density alone do not provide that map.

Close approximation precedents include Bizeul and Klartag's treatment of critical exponential weights using a sech basis and complex analysis [14] and Koehler–Wu's Fourier-tail plus strip-tanh $L^2(\mu)$ bound under an exponential moment [15, Theorem 4.7]. Hua–Yang [16, Section 5] already use Gevrey frequency buffers to absorb large amplitude and pay nuclear localization tails. Their inspected conclusions are not the original theta joint derivative and absolute-potential assertion proved here; our argument supplies that additional map explicitly.

We use the standard entire normalization $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Put $G = F'^2 - FF''$ and $V = G/F^2$ off the real zeros. We use

$$E(P) = \frac{1}{4\pi}\int_{\mathbb{R}} |(FP)'|^2, \qquad Q(P) = \frac{1}{8\pi}\int_{\mathbb{R}} GP^2, \qquad D(P) = E(P) - Q(P). \tag{1}$$

For a raw target u write $E(u) = (4\pi)^{-1}\int |u'|^2$, $Q(u) = (8\pi)^{-1}\int V|u|^2$ and

$$\|u\|_*^2 = \int_{\mathbb{R}} (|u'|^2 + (1+|V|)|u|^2). \tag{2}$$

Values assigned to V at isolated real zeros do not affect these integrals. For $u = FP$ these definitions agree with (1). The absolute potential is essential; controlling only a signed integral is insufficient. All polynomial spaces below are real, and a rational basis means that its basis polynomials have rational monomial coefficients.

Theorem 1.1. *Fix $0 < \theta < 1$. There are $c, C > 0$ and N_0 such that, for every integer $N \geq N_0$ and each parity, two possibly different polynomial spaces of degree at most N have dimension at least*

$$C\exp(c\sqrt{\log N})\sqrt{\log N}. \tag{3}$$

Each has a rational basis. On the entire nonzero span of the first, $Q/E \geq 2-\theta$; on that of the second, $0 < Q/E \leq \theta$. Consequently D has negative and positive index at least (3) in each parity finite section.

All geometric choices and constants are fixed before N increases. The cutoff order is chosen by an explicit rule with uniform constants; the MP circle order stays fixed. The onset can depend on the fixed choices. Both channels have positive original Q; no same-state complementary sign is inferred. The quotient bound and full-span quantifier are stronger than separate basis-vector tests.

2 The complete local factor and classical populations

Set $a_* = \pi/8$. The xi formula, Stirling's formula and polynomial zeta bounds on fixed vertical strips give, for every fixed b_0,

$$|F(x+iy)| \leq C_{b_0}(1+|x|)^{p_{b_0}}e^{-a_*|x|}, \qquad |y| \leq b_0. \tag{4}$$

For example the continuation formula $\zeta(s) = s/(s-1) - s\int_1^\infty \{u\}u^{-s-1}\,du$ first gives polynomial bounds in $\Re s > 0$; the functional equation handles the remaining fixed strip. The xi prefactors remove the apparent poles. Cauchy's estimate in fixed disks therefore gives a fixed integer $p \geq 1$ with

$$|F(x)| + |F'(x)| + \sqrt{|G(x)|} \leq C(1+|x|)^p e^{-a_*|x|}. \tag{5}$$

These standard estimates are uniform through real zeros; see [6, Chapters II and IX]. We include the local product argument from [2] because the complex version is needed in our receiver.

The following lemma includes every nonreal zero in its local disk. This is necessary: the individual nonreal pole terms need not be positive on the real line.

Lemma 2.1 (Uniform local factor)**.** *There are fixed positive constants C_0, C_1, C_2 with the following property for all sufficiently large T. Set $K = \lceil C_0 \log(T+40) \rceil \geq 2$. For any real $c \in [10, T]$ choose any $b_c \in (8, 9)$ for which the circle $|z - c| = b_c$ contains no zero, and let Z_c be the multiset of all zeros inside. Then $\#Z_c \leq K$, and on the real interval $|x - c| \leq 1$, away from zeros,*

$$V(x) = \sum_{\rho \in Z_c} \frac{1}{(x-\rho)^2} + R_c(x), \qquad |R_c(x)| \leq C_1 K. \tag{6}$$

On the same interval, for any positive numbers $d_\rho \leq |x - \rho|$,

$$|F(x)| \geq e^{-\pi c/8} (T+40)^{-C_2} e^{-C_2 K} \prod_{\rho \in Z_c} \min(1, d_\rho). \tag{7}$$

All products and sums count multiplicities.

Proof. The Riemann–von Mangoldt formula with its $O(\log T)$ error [6, Chapter IX] bounds the number of all zeros, with multiplicities, in every fixed-width real window up to height $T + 20$ by $C \log(T + 40)$. All zeros have imaginary part of magnitude less than one. This bounds the zeros in every disk used in the statement, after fixing C_0.

Normalize $f_c(w) = e^{\pi c/8} F(c + w)$. Formula (4), used on the wider fixed strip needed for $|w| \leq 20$, implies $|f_c(w)| \leq C(T + 40)^p$ there. The off-axis point $w = -3i$ is a uniform anchor: its zeta argument is $2 + ic/2$, and

$$|\zeta(2 + ic/2)| \geq \frac{1}{\zeta(2)}, \qquad |\Gamma(1 + ic/4)|^2 = \frac{\pi c/4}{\sinh(\pi c/4)}.$$

The first bound follows by taking absolute values in the absolutely convergent reciprocal Euler series. The xi prefactors and the last identity give $|f_c(-3i)| \geq a_3 > 0$ for $c \geq 10$.

In the disk $|w| < b = b_c$, divide f_c by the full finite disk Blaschke product

$$B_c(w) = \prod_{\rho \in Z_c} \frac{b(w - (\rho - c))}{b^2 - \overline{(\rho - c)} w},$$

where unimodular factors may be inserted without changing any bound. The quotient $H_c = f_c / B_c$ is holomorphic and zero-free in that disk. Since $|B_c| = 1$ on its boundary, the maximum principle gives $|H_c| \leq C(T + 40)^p$ inside. Also $|B_c(-3i)| \leq 1$, so $|H_c(-3i)| \geq a_3$. For a suitably enlarged polynomial bound M, $U = \log M - \log |H_c|$ is positive harmonic. Harnack's inequality first compares $-3i$ with zero, then zero with $|w| \leq 2$. All comparison constants are uniform for $8 < b < 9$. Thus

$$0 \leq U(w) \leq C \log(T + 40), \qquad |H_c(w)| \geq e^{-C \log(T+40)} \quad (|w| \leq 2).$$

The harmonic derivative estimate on concentric disks (equivalently Borel–Caratheodory followed by Cauchy's estimate for a holomorphic logarithm) gives $|(\log H_c)''(w)| \leq C \log(T + 40)$ for $|w| \leq 1$.

Write $\alpha = \rho - c$. The second logarithmic derivative of its Blaschke factor is

$$-\frac{1}{(w - \alpha)^2} + \frac{\overline{\alpha}^2}{(b^2 - \overline{\alpha} w)^2}.$$

The last term has magnitude at most $(b - 1)^{-2} \leq 1/49$ on $|w| \leq 1$. Since $V = -(\log F)''$ locally away from zeros, summing these identities proves (6). For real $|w| \leq 1$, each factor also satisfies

$$\left| \frac{b(w - \alpha)}{b^2 - \overline{\alpha} w} \right| \geq \frac{|w - \alpha|}{b + 1} > \frac{|w - \alpha|}{10}.$$

Multiply these inequalities and the lower bound for H_c, and undo the normalization $e^{\pi c/8}$. The factors $10^{-\#Z_c}$ are absorbed in e^{-C_2K}, while the quotient loss is a power of $T+40$. Clipping each assigned distance at one only weakens the bound. This proves (7) uniformly even when roots approach one another or the chosen boundary circle. □

2.1 A simultaneous microscopic selection

Let $N_F(t) = N_\zeta(t/2)$ count all zeros by real coordinate, with multiplicity. This is the correct scaling because a zeta zero $\beta+i\gamma$ corresponds to an F zero $2\gamma+i(1-2\beta)$. The classical input from Fujii [4, Lemma 1, p. 742] is only the fixed fourth increment moment. For $U \asymp T$ and $\delta = \ell/[2\log(T+40)]$, with fixed $\ell > 0$, it implies

$$\int_U^{2U} |S_\zeta(t+\delta) - S_\zeta(t)|^4 \, dt \le C_\ell U.$$

Indeed $\log(3+\delta\log U)$ is bounded in that lemma, so both its fixed-moment main term and error are $O(U)$. Cauchy–Schwarz gives the second moment. The unconditional moment upper bound is also explicitly stated in [5, Theorem 4]. Riemann–von Mangoldt, a fixed finite dyadic cover and $t = 2u$ now give, for fixed $a \in (0,1)$,

$$\int_{aT/2}^{T+5} [N_F(t+\ell h) - N_F(t)]^2 \, dt \le C_{a,\ell} T, \qquad h = \frac{1}{\log(T+40)}. \tag{8}$$

Its smooth main-term increment is $O_\ell(1)$; zero endpoints form a null set. No pair-correlation hypothesis or growing moment is used.

The positive-proportion simple critical-line count [3, Theorem 1.1] supplies $b_0 T\log T$ distinct simple real seeds below T. Choose $a > 0$ small enough that the all-zero count below aT is at most half that quantity eventually. This subtraction, rather than an unsupported dyadic lower count, leaves bT/h seeds in $[aT, T]$. End bins cost $O(\log T)$ and can be discarded.

Partition into bins $I_j = [jh, (j+1)h)$. Let n_j count every zero with real coordinate in I_j, including complex zeros and multiplicities, and let $s_j \le n_j$ count the simple real seeds in central bins. Applying (8) with $\ell = 3$ on the disjoint intervals $((j-1)h, jh)$ yields

$$\sum_{I_j \cap [aT-4, T+4] \neq \varnothing} n_j^2 \le CT/h. \tag{9}$$

This enlarged range covers all roots of distance less than two from a central seed.

Define $q_j = \sum_{|l|\le 2} n_{j+l}$ and the symmetric sequence

$$w_l = \begin{cases} \log(2/h), & |l| \le 2, \\ \log_+(2/[(|l|-1)h]), & |l| \ge 3. \end{cases}$$

Its finite support and $\sum_l w_l \le C/h$ follow by comparison with $h^{-1}\int_0^2 \log(2/x)\,dx$. Truncate n_j to the enlarged range when forming convolutions and put $H_j = \sum_l w_l n_{j+l}$ and $L_j = \sum_l n_{j+l}/(1+l^2)$. Nonnegative summation and Cauchy–Schwarz in (9) give

$$\sum_{j \text{ central}} q_j^2 \le CT/h, \qquad \sum_{j \text{ central}} s_j H_j \le CT/h^2, \qquad \sum_{j \text{ central}} s_j L_j \le CT/h. \tag{10}$$

For the last inequality, each shifted inner product is at most CT/h and $\sum_l (1+l^2)^{-1} < \infty$. The first uses only the five neighboring shifts. The second uses the integrable logarithmic kernel. All shifted factors stay in the finite enlarged range; no whole-axis square-count sum is presumed.

Choose fixed J, B_1, B_2 large enough. The union of bins with $q_j > J$, $H_j > B_1/h$ or $L_j > B_2$ loses less than half the seeds: the respective losses are bounded by $CT/(Jh)$, $CT/(B_1 h)$ and $CT/(B_2 h)$. Each retained bin has at most J seeds. Choose one residue class modulo four and one seed in each occupied retained bin. There are $M \asymp T/h$ selected seeds separated by at least $3h$, and every inner disk $|\rho - r| < h$ contains at most J roots.

For each retained seed apply Lemma 2.1 with $c = r$. Split its complete macroscopic root multiset into inner $|\alpha| < h$ and outer $|\alpha| \geq h$, where $\alpha = \rho - r$. For $h \leq |\alpha| < 2$, a root in bin $j + l$ satisfies $|\alpha|^{-2} \leq Ch^{-2}/(1 + l^2)$: use the outer condition for $|l| \leq 2$, and $|\alpha| \geq (|l| - 1)h$ otherwise. Roots of distance at least two instead contribute at most $K/4$. Thus the SAME seeds obey

$$\sum_{\text{outer}} |\alpha|^{-2} \leq C_* h^{-2}, \qquad \sum_{\text{outer}} \log_+(2/|\alpha|) \leq B_1/h. \tag{11}$$

This treats the full macroscopic disk without extending (9) beyond its proved coordinate range.

3 Relative root budgets and localized channels

Fix an error $\epsilon > 0$ to be chosen from the desired quotient margins. Choose a fixed large $D_* > 1$ and set

$$t_0 = h/D_*, \qquad S_0 = -\log t_0.$$

For $0 < t \leq t_0$, the outer poles in (6) cost at most $4C_*/D_*^2$ after multiplication by t^2. The remainder costs CKh^2/D_*^2. Both are below ϵ for fixed large D_* and large T.

For each nonzero inner root set $\lambda_\alpha = -\log|\alpha|$; for the seed set $\lambda_0 = \infty$. Take fixed large constants A_*, B_* and delete

$$[\lambda_\alpha - \log A_*, \lambda_\alpha + \log B_* + \tfrac{1}{2}\log J].$$

Outside these at most J transition intervals, each inner root has either $|\alpha| \leq t/A_*$ or $|\alpha| \geq B_*\sqrt{J}\,t$, where $t = e^{-s}$. The seed is always interior. For an interior root, $|(1 - \alpha/t)^{-2} - 1| \leq C/A_*$; for the remaining inner roots, the absolute total is at most $4/B_*^2$. Conjugation makes the complete sum real. Choose A_*, B_*, D_* so that on every good component

$$(1 - 2\epsilon)\nu \leq t^2 V(r + t) \leq (1 + 2\epsilon)\nu, \qquad 1 \leq \nu \leq J. \tag{12}$$

Put $\mathcal{B}(s) = \sum_{\text{inner}} \min(s, \lambda_\alpha)$. It is continuous with $1 \leq \mathcal{B}' \leq J$ almost everywhere; on a good component $\mathcal{B}' = \nu$. For any relative budget $L > 0$ choose the unique S satisfying $\mathcal{B}(S) - \mathcal{B}(S_0) = L$. Then

$$S_0 + L/J \leq S \leq S_0 + L, \qquad e^S \leq D_* h^{-1} e^L. \tag{13}$$

The deleted intervals cost at most a fixed $J^2 w_J$ in weighted length. The good weighted length is therefore at least $L - J^2 w_J$, in at most $J + 1$ components.

Let $\chi \geq 0$ be any one fixed nonzero flat smooth cutoff in $(0, 1)$, and set $I_0 = \int \chi^2$, $I_1 = \int |\chi'|^2$. In a good component pack caps of length $\ell_\nu = \ell/\sqrt{\nu}$, where $\ell > 0$ is fixed. On a cap starting at a_c define

$$v_c(s) = \left(\frac{4\pi\ell_\nu}{I_1 + I_0\ell_\nu^2/4}\right)^{1/2} \chi((s - a_c)/\ell_\nu), \qquad u(r + t) = \sqrt{t}\, v_c(-\log t), \tag{14}$$

and extend by zero. Direct substitution gives

$$4\pi E(u) = \int (|v_c'|^2 + |v_c|^2/4) = 4\pi, \tag{15}$$

$$8\pi Q(u) = \int t^2 V(r + t)|v_c|^2. \tag{16}$$

The boundary cross term vanishes. For the pure pole the quotient is

$$q_\nu = \frac{2\nu}{1 + 4(I_1/I_0)\nu/\ell^2}. \tag{17}$$

Choose ℓ large and ϵ small to obtain $Q/E \geq 2 - \theta/4$, uniformly in $1 \leq \nu \leq J$. Alternatively choose ℓ small and then ϵ small to obtain $q_{\min} \leq Q/E \leq \theta/8$ for one fixed $q_{\min} > 0$. Both choices have uniformly bounded $\|u\|_*$; the derivative part is normalized and the potential part is bounded by (12), while $\int |u|^2 \leq C t_0^2 \int |v_c|^2$.

The packing count is at least $(L - J^2 w_J)/(\ell\sqrt{J}) - (J+1)$. Hence each selected seed supplies at least cL caps once $L \geq L_0$, with L_0 independent of T. Taking exactly a fixed proportional number and using (13) gives, for either family,

$$d \asymp T \log T \, L. \tag{18}$$

Distinct caps have disjoint supports; the seed spacing exceeds $3h$ and each support is within $t_0 < h$ on the right of its seed. Reflect and divide by $\sqrt{2}$ to obtain either parity. The raw E Gram is exactly the identity, the raw Q Gram is diagonal with the stated margins, and every raw joint norm is bounded by a fixed M_0. No cross Gram between the two different families is required.

3.1 Complex division on the same selected caps

Let $g(z) = F(z)\cosh(a_* z)$. A fixed sufficiently small disk $|z - (r+t)| \leq c_\theta t$ around a good point avoids every actual local root. Good-scale separation gives $|z - \rho| \geq c \max(t, |\rho - r|)$ for inner roots, and $|z - \rho| \geq |\rho - r|/2$ for outer roots. The Harnack lower bound for the same H_c in the local-factor proof holds on the entire fixed smaller complex disk, not merely its real diameter. Its Blaschke denominators cost e^{CK}. The outer product is bounded below by $\exp(-CB_1/h)$ by (11); the inner product is bounded below by $\exp[-CJ - \mathcal{B}(s)]$. Since $\mathcal{B}(s) \leq \mathcal{B}(S_0) + L \leq JS_0 + L$, and J is fixed, all initial microscopic product loss is absorbed into a fixed power of T. On these large positive-real disks $|\cosh(a_* z)| \asymp e^{a_* r}$; its second exponential is negligible uniformly. The reflected disks have the same bound by parity. Therefore

$$\sup_{|z-(r+t)| \leq c_\theta t} |1/g(z)| \leq T^C e^{CL}. \tag{19}$$

The imaginary-axis zeros of cosh are outside these disks. Cauchy applied to the reciprocal itself gives

$$|(1/g)^{(n)}(r+t)| \leq T^C e^{CL} n! [Ch^{-1} e^L]^n. \tag{20}$$

All roots outside the macroscopic disk remain in H_c; none was truncated. The amplitude is paid once, independently of the differentiation order.

4 The critical MP receiver in the full joint norm

The critical sech-basis/strip method is closely related to [14, 8]; our return estimate additionally pays the original differential joint norm. This section separates the critical transform from the choice of compact cutoff. Write $a = a_*$, $H_M = \pi/(2a)$ and

$$\omega(x) = \frac{1}{H_M \cosh(\pi x/H_M)}, \qquad q_n(x) = P_n^{(1/2)}(x/H_M; \pi/2).$$

These real MP polynomials form an orthonormal basis in $L^2(\omega)$. Their classical generating function and recurrence [7] specialize to

$$\sum_{n\geq 0} q_n(x)z^n = (1-iz)^{-1/2+ix/H_M}(1+iz)^{-1/2-ix/H_M}, \tag{21}$$

$$xq_n = \frac{H_M}{2}[(n+1)q_{n+1} + nq_{n-1}], \qquad q_{-1} = 0. \tag{22}$$

The normalization is such that $q_0 = 1$ and $q_1 = 2x/H_M$. Completeness also follows directly: if $f \in L^2(\omega)$ is orthogonal to all polynomials, then $f\omega \in L^1$ has an analytic Fourier transform on a nonempty strip, by Cauchy–Schwarz and the exponential moments of ω. All its derivatives at zero vanish, so analytic uniqueness and Fourier injectivity give $f = 0$.

Lemma 4.1. *For the fixed p in (5), $\|Fq_n\|_* \leq C(n+1)^p$ for every $n \geq 0$.*

Proof. Repeated use of (22) gives $\|x^j q_n\|_{L^2(\omega)} \leq C_j(n+1)^j$. Differentiating (21) in x gives the finite convolution

$$q_n' = \frac{2}{H_M} \sum_{\substack{1\leq l\leq n \\ l \text{ odd}}} \frac{(-1)^{(l-1)/2}}{l} q_{n-l}.$$

After multiplication by x^j, each summand occupies only the basis indices within j of $n-l$. Each output index has at most $2j+1$ contributors. Cauchy–Schwarz with this bounded overlap, followed by $\sum l^{-2} < \infty$, yields

$$\left\|x^j q_n'\right\|^2_{L^2(\omega)} \leq C_j(n+1)^{2j}.$$

Since $\omega \asymp e^{-2a|x|}$, the product rule and (5) prove the claim. At real zeros the potential term is paid by $|V|F^2 = |G|$ almost everywhere. No pointwise division bound is needed in this step. □

For a function e define the fixed weighted Sobolev norm

$$|e|_{p,1} = \sum_{j=0}^{p} (\left\|x^j e\right\|_2 + \left\|x^j e'\right\|_2).$$

The product rule and (5) imply

$$\|F\cosh(ax)e\|_* \leq C|e|_{p,1}. \tag{23}$$

In particular $|V|F^2\cosh^2(ax) = |G|\cosh^2(ax)$ has polynomial growth globally. The derivative term includes $F'\cosh$ and $aF\sinh$.

Proposition 4.2 (Bandlimited critical receiver)**.** *Let v be real and smooth, supported in $[-T,T]$, $T \geq 1$, with $\|v\|_2 \leq B$, $B \geq 1$. Fix a real even smooth frequency cutoff χ_0, equal to one on $[-1,1]$ and zero outside $[-2,2]$. For $H \geq 1$ put*

$$b_H(k) = \frac{1}{2\pi}\chi_0(k/H)\widehat{v}(k), \quad v_H(x) = \int b_H(k)e^{ikx}\,dk, \qquad \widehat{v}(k) = \int v(x)e^{-ikx}\,dx.$$

For $0 < \eta < 1$, if $|v - v_H|_{p,1} < \eta/(2C)$, there exists a real polynomial P, preserving the parity of v, with

$$\|FP - F\cosh(ax)v\|_* < \eta, \qquad \log(\deg P + 2) \leq C[H + \log(B(T+2)/\eta + 2)]. \tag{24}$$

For any finite family, rational rounding preserves degree and parity with any additional positive joint-error allowance.

Proof. We give the boundary coefficient argument explicitly. Integration of (21) against $e^{ikx}\omega(x)$ gives

$$\int e^{ikx} q_n(x)\omega(x)\,dx = \operatorname{sech}(H_M k/2)[i\tanh(H_M k/2)]^n.$$

For each fixed n this identity continues to $|\Im k| < \pi/H_M$. Thus the shifts $k \mp ia$ are strictly inside the strip. Fubini for compact k support is justified by weighted L^1 integrability. The MP coefficients of $f_H = \cosh(ax)v_H \in L^2(\omega)$ are

$$c_n = \frac{1}{2}\sum_{\varepsilon=\pm1}\int b_H(k)A_\varepsilon(k)\zeta_\varepsilon(k)^n\,dk, \tag{25}$$

where

$$A_\varepsilon = \operatorname{sech}(H_M k/2 - \varepsilon i\pi/4), \qquad \zeta_\varepsilon = \varepsilon\operatorname{sech}(H_M k) + i\tanh(H_M k).$$

Each phase lies exactly on the unit circle. Set $\vartheta_+ = \arcsin(\tanh(H_M k))$ and $\vartheta_- = \pi - \vartheta_+$. These parametrize opposite half circles, with one reversed orientation, and

$$|\vartheta_\varepsilon'| = H_M\operatorname{sech}(H_M k), \qquad |A_\varepsilon|^2 = 2\operatorname{sech}(H_M k).$$

On the two arcs put $\psi(\vartheta_\varepsilon(k)) = b_H(k)A_\varepsilon(k)/(2|\vartheta_\varepsilon'(k)|)$, and extend by zero. The cutoff is flat at the finite arc endpoints. Equation (25) becomes $c_n = \int \psi(\vartheta)e^{in\vartheta}\,d\vartheta$. The order-zero change of variables cancels exactly: $|A_\varepsilon|^2/|\vartheta_\varepsilon'| = 2/H_M$. Each fixed-order circle derivative introduces at most a factor $C_s\operatorname{sech}(H_M k)^{-1}$. Logarithmic k derivatives of the remaining amplitude and Jacobian are bounded at every fixed order. Consequently

$$\|\psi\|_{H^s(\mathbb{T})} \le C_s e^{2sH_M H}\,\|b_H\|_{H^s(\mathbb{R})} \le C_s e^{2sH_M H} B(1+T)^s. \tag{26}$$

The last inequality is Plancherel applied to the first s spatial moments of the compact v; derivatives of $\chi_0(k/H)$ cost $H^{-l} \le 1$. Parseval for the complete integer Fourier series and Cauchy–Schwarz give, for fixed $s > p + 1/2$,

$$\sum_{n>n_0}|c_n|(n+1)^p \le C_s n_0^{p-s+1/2} e^{2sH_M H} B(1+T)^s. \tag{27}$$

Circle coefficients retain their signs in the expansion.

Choose the fixed integer $s = p + 2$. Lemma 4.1 shows that $F\sum_{n\le n_0} c_n q_n$ converges absolutely in the original joint norm, with tail bounded by the right side of (27), whose power is $-3/2$. MP completeness identifies the polynomial partial sums with f_H in $L^2(\omega)$. On every compact real interval multiplication by F is bounded into ordinary L^2; hence the same partial sums converge there to Ff_H. The global joint limit must therefore agree locally, and thus almost everywhere globally, with Ff_H. Closedness of the derivative and weighted L^2 norms identifies all parts of the joint limit. Also Ff_H belongs to that norm directly by (23), since v_H is Schwartz. We do not assume that multiplication by F is bounded from $L^2(\omega)$ to unweighted global L^2; its polynomial factor would invalidate that shortcut.

Taking n_0 to be the ceiling of a sufficiently large exponential in $H + \log(B(T+2)/\eta + 2)$ makes the joint tail below $\eta/2$. Equation (23) pays the other half, proving (24). A real target has real MP coefficients; the even cutoff and parity of q_n preserve either parity. Finally all $\|Fx^j\|_*$ are finite by (5). Rounding the finitely many correct-parity monomial coefficients therefore gives rational coefficients with arbitrary positive remaining error, without raising degree. □

5 Finite-order cutoffs with uniform cap energies

The cutoff may depend on the requested finite approximation order. This is the classical Ehrenpreis–Hörmander cutoff-sequence principle; see, for example, the beginning of [17, Section 3]. It is different from requiring one compact function to have analytic bounds at every order. We give the construction and all uniform low-order bounds, since the two theta channels need those bounds.

Choose once a nonnegative smooth probability density φ_0 supported in $[-1/8, 1/8]$, for example the normalized function $\exp[-1/(1-64x^2)]$ there, extended by zero. Put $\delta = 1/4$. For each integer $m \geq 1$ let $w_m = \delta/m$, let H_w be the uniform probability density on $[-w/2, w/2]$, and define

$$\chi_m(x) = (\varphi_0 * H_{w_m}^{*m})(x - 1/2). \tag{28}$$

Every χ_m is nonnegative, smooth, flat at its support endpoints, has mass one, and is supported in $[1/4, 3/4]$.

Lemma 5.1. *Writing $I_{0,m} = \int \chi_m^2$ and $I_{1,m} = \int |\chi_m'|^2$, there are fixed $0 < C_{\min} \leq C_{\max} < \infty$ with*

$$2 \leq I_{0,m} \leq \|\varphi_0\|_2^2, \qquad I_{1,m} \leq \|\varphi_0'\|_2^2, \qquad C_{\min} = \pi^2 \leq I_{1,m}/I_{0,m} \leq C_{\max}. \tag{29}$$

For $0 \leq k \leq m$,

$$\left\|\chi_m^{(k)}\right\|_\infty \leq \|\varphi_0\|_\infty \, (2m/\delta)^k. \tag{30}$$

Proof. All smoothing factors are probabilities. Young's inequality proves the upper L^2 and derivative bounds, and Cauchy–Schwarz on the common support of length $1/2$ proves $I_{0,m} \geq 2$. Poincaré's inequality on $(0,1)$ gives $I_{1,m} \geq \pi^2 I_{0,m}$. Differentiate k distinct box factors as signed measures; each derivative has total variation $2/w_m$. Leave φ_0 as the bounded density and the remaining factors as probabilities. This proves (30), without any bound on derivatives above order m. □

Use χ_m in (14). By (29), one fixed long length ℓ_{long} and one fixed short length ℓ_{short} work for every m in (17). The complete long-cap quotient is at least $2 - \theta/4$. For the short caps choose $\ell_{\text{short}}^2/(2C_{\min}) \leq \theta/16$ and ϵ small; then

$$q_{\min} \leq Q/E \leq \theta/8, \qquad q_{\min} = \frac{(1-2\epsilon)2}{1 + 4C_{\max}/\ell_{\text{short}}^2} > 0.$$

The normalization factor in (14) is uniformly bounded by (29); all raw joint norms are at most one fixed M_0. Thus the fixed packing threshold L_0, cap count and both margins are independent of m.

Lemma 5.2 (Division at a controlled finite order)**.** *Let $v = u/[F\cosh(a_* x)]$ for any such cap or its parity reflection. Put $B_0 = T^C e^{CL}$ and $A_0 = Ch^{-1}e^L \geq 1$. For every fixed $j \leq p+1$, uniformly for integers $m \geq 1$,*

$$\|v\|_\infty + \|v\|_2 \leq CB_0, \tag{31}$$

$$\left\|D^m(x^j v)\right\|_\infty \leq C_j B_0 (1+T)^j m! (CA_0)^m. \tag{32}$$

Support measure is bounded by a fixed constant and support lies in $[-CT, CT]$. Constants do not depend on m, T, L.

Proof. For the logarithmic coordinate $\varphi(t) = (-\log t - a_c)/\ell_\nu$, $|\varphi^{(j)}|/j! \le K_0 t^{-j}$ with fixed K_0. Faà di Bruno, (30), and the ordered-composition identity

$$\sum_{\substack{\sum k_j = k \\ \sum j k_j = n}} \frac{k!}{\prod k_j!} = \binom{n-1}{k-1}$$

give, for $1 \le n \le m$,

$$\begin{aligned} |D^n[\chi_m(\varphi(t))]| &\le Cn! t^{-n} \sum_{k=1}^{n} \frac{(CmK_0)^k}{k!} \binom{n-1}{k-1} \\ &\le Cn!(2/t)^n e^{Cm}. \end{aligned}$$

The denominator $k!$ is essential. We do not replace e^{Cm} by C^n when n is small. Multiply by the analytic reciprocal and $\sqrt{t}$, using (20) and the analytic branch on the local disks of radius $c_\theta t$. Leibniz's exact factorial cancellation yields

$$\left\| v^{(n)} \right\|_\infty \le CB_0 n!(CA_0)^n e^{Cm}, \qquad 0 \le n \le m.$$

The sharper zeroth bound (31) follows separately from $\|\chi_m\|_\infty \le \|\varphi_0\|_\infty$, bounded support measure, and (19). At order m, Leibniz with x^j has only $j+1$ terms. The factor $\binom{m}{l}(m-l)!/m! = 1/l!$ and $A_0 \ge 1$ show that its e^{Cm} loss is absorbed into one fixed constant to power m. This proves (32). The energy prefactor is uniformly bounded, and zero extensions have flat traces. □

Proposition 5.3 (Finite-order joint receiver)**.** *For the targets of Lemma 5.2 and $0 < \eta < 1$, choose*

$$m = \left\lceil C \left[1 + \log \frac{CB_0(T+2)^{p+1} A_0^{3/2}}{\eta} \right] \right\rceil, \qquad H = KA_0 m, \tag{33}$$

with fixed sufficiently large C, K, so $m \ge 3$. There exists a same-parity real polynomial P with $\|FP - u\|_* < \eta$ and a sufficient degree n satisfying

$$\log(n+2) \le CA_0 \left[1 + \log \frac{CB_0(T+2)^{p+1} A_0^{3/2}}{\eta} \right]. \tag{34}$$

Finite rational rounding can be included in the tolerance.

Proof. Integrating by parts m times, with flat traces, gives for fixed $j \le p+1$ and $k \ne 0$,

$$|D_k^j \widehat{v}(k)| \le C_j B_0 (1+T)^j m!(CA_0)^m |k|^{-m}.$$

Use the smooth frequency truncation of Proposition 4.2. Plancherel applied to the needed frequency derivatives of both the tail and k times the tail yields

$$|v - v_H|_{p,1} \le CB_0(1+T)^p m!(CA_0)^m \frac{H^{3/2-m}}{\sqrt{2m-3}}. \tag{35}$$

Indeed the main integral is $\int_H^\infty k^{2-2m} dk = H^{3-2m}/(2m-3)$. Cutoff derivatives have only the fixed orders through p, cost $H^{-l} \le 1$, and occur at $|k| \ge H$; differentiating the additional factor k only improves the bound. This controls the entire weighted value-plus-derivative norm on $\mathbb{R}$, not only a

compact interval or unweighted value error. With $H = KA_0m$ and $m! \le m^m$, the right side is at most

$$CB_0(1+T)^p A_0^{3/2}(Km)^{3/2}(C/K)^m \le CB_0(1+T)^p A_0^{3/2} e^{-cm}$$

after increasing fixed K and the lower threshold for m. The rule (33), with all constants independent of m, makes (23) pay less than half the joint error.

For the MP return use the direct low-order bound (31). Its fixed circle order $s = p+2$ requires only spatial moments, bounded by $B_0(1+T)^s$, not derivative control above m. Proposition 4.2 gives a sufficient log degree $C[H+\log(T+2)+\log(B_0/\eta+2)]$; substituting (33) proves (34). Reality, parity, local identification and global absolute-potential convergence are all retained. Reserve additional positive slack for the same-degree rational rounding. □

The finite order m explicitly depends on the required accuracy; its constants have been controlled uniformly above. The MP circle order remains fixed. No one compact analytic function or unspecified growing-order Jackson constant is used.

6 Whole Gram transfer and the square-root logarithmic scale

Theorem 6.1 (Finite-resource theta localization)**.** *Fix $0 < \theta < 1$ and one relative budget $L \ge L_0$ sufficiently large for both cap lengths. For all sufficiently large T, each parity has two possibly different rational-basis polynomial spaces of dimension $d \asymp T\log T$ and degree at most n, preserving respectively $Q/E \ge 2-\theta$ and $0 < Q/E \le \theta$ on their entire nonzero spans. A sufficient common degree satisfies*

$$\log(n+2) \le C(\log T)^2. \tag{36}$$

Constants depend on the fixed margin and choices, not T.

Proof. First fix θ, the base φ_0, uniform long and short cap lengths, geometric thresholds and one $L \ge L_0$. The geometric slots, their count $d \asymp T\log T$, raw E/Q margins and joint bound M_0 are independent of m. Choose one common tolerance

$$\eta \le \frac{c_\theta}{d(1+M_0)^2},$$

including the fixed positive $q_{\min}$ in the short-family constant. This choice uses only uniform bounds, not a target that has yet to be chosen. Now select m by (33), fill the slots with χ_m, and apply Proposition 5.3. This order of choices is noncircular. Reserve half the error for rational monomial rounding.

The raw E Gram is I and the raw Q Gram is diagonal with the proved margins. Cauchy–Schwarz in the derivative and absolute-potential parts bounds each polarized E or Q entry change by $C(2M_0\eta+\eta^2)$. The operator norm of each full error matrix is at most d times its largest entry. Choose c_θ small enough to preserve strict positivity of E, independence and the desired quotient for every real coefficient vector. In the long family start from $2-\theta/4$; in the short family preserve half the positive minimum and the upper margin $\theta/8$. This pays every mixed entry; it is not a basis-vector Rayleigh test. No mutual cross Gram between the two different families is required.

For fixed L, $A_0 = O(\log T)$, $\log B_0 = O(\log T)$ and $\log(1/\eta) = O(\log T)$. Thus the explicitly selected $m = O(\log T)$, and (34) gives (36). All constants in the varying-order step have already been paid. All infinite roots and global tails are retained in the geometric and joint receivers. □

Proof of Theorem 1.1. Put $X = \log N$ and choose $\log T = c\sqrt{X}$, with one fixed $c > 0$ sufficiently small. The right side of (36) is at most $X/2$ eventually, after absorbing all fixed constants. Integer roundings and the common constructed degree therefore fit below N for every sufficiently large integer N, not only along a subsequence. The count $d \geq CT \log T$ gives (3) for both channel families, separately in either parity. □

6.1 A necessary rank for a specified contraction scheme

The matrix E_N in either parity degree-N section is positive definite: $(FP)' = 0$ makes FP constant, decay forces that constant zero, and then $P = 0$. Put $K_N = E_N^{-1/2} D_N E_N^{-1/2}$.

Corollary 6.2. *Fix $0 < c_0 < 1$. If real α_N and symmetric C_N satisfy*

$$\|K_N - \alpha_N I + C_N\|_{\mathrm{op}} \leq c_0, \tag{37}$$

then $\operatorname{rank} C_N \geq C \exp(c\sqrt{\log N})\sqrt{\log N}$ *for all sufficiently large N, with constants depending on c_0. The same assertion holds in the full degree section even when the correction mixes parities.*

Proof. Choose fixed $\theta < 1 - c_0$. The two Euclideanized spaces have $D/E \leq -1+\theta$ and $D/E \geq 1-\theta$. If the rank were smaller than both dimensions, each space would meet $\ker C_N$ nontrivially. On their unit vectors, (37) forces respectively $\alpha_N \leq c_0 - 1 + \theta$ and $\alpha_N \geq 1 - \theta - c_0$, a contradiction. Congruence preserves rank for a correction expressed in original coordinates. In the full section embed either parity's spaces and use the same intersection argument. No cross Gram between the two spaces and no definiteness of C_N are required. □

This is a necessary condition for the architecture (37), not a sufficient correction and not an obstruction to every positivity proof or algorithm.

7 Reproducibility and limits

The supplement `verify_weighted.py` uses only Python's standard library and exact fractions. It checks MP Gram values against the original Euler moments, derivative coefficients, weighted norms, pole-bin majorants and finite convolution bounds. It does not numerically sample xi zeros. These finite controls check algebraic inputs. The proofs above provide those deductions.

The dimension scale exceeds every fixed $\exp((\log N)^\delta)$ for $\delta < 1/2$ and every fixed logarithmic power. Fixed nearanalytic cutoffs can approach this scale with logarithmic losses; the order-dependent family above avoids those losses while keeping both raw energies uniform. The cutoff's selected finite order grows explicitly as $O(\log T)$; its higher derivatives are not given uniform analytic bounds. The classical MP circle order, geometric thresholds and all constants stay fixed before N increases. No optimal rate or constant, positive-proportion-of-degree index, coefficient-bit bound, feasible onset or runtime estimate follows. All local complex roots and exterior analytic roots are retained. Both constructed original Q channels are strictly positive.

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AI tools assisted with mathematical exploration, drafting, symbolic and exact-arithmetic checks, source comparison and internal proof review. Scott Schubr is the author and assumes responsibility for the manuscript.

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