Audibility of all odd blind modes in rapidly twisted polar-strip tubes

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Luminary Oddities

Abstract

The leading Weyl profile of a rapidly twisted Euclidean tube can hide all odd logarithmic harmonics of its inverse twisting speed. For one fixed connected polar section, we prove that the next heat profile detects the entire odd lattice. The actual-tube two-term expansion is proved internally, retaining the connected aperture contribution. A positive-root matrix argument and controlled aperture limits represent the full connected-minus-cut relative heat trace as the Laplace transform of a positive measure. This mechanism extends to every junction of two straight halfstrips with nested bounded transverse intervals, arbitrary widths and offset: its full relative trace is completely monotone. For the fixed polar section, an analytic mass bound and Euler-product domination give two-sided Gamma-scale bounds for every odd Mellin multiplier. Thus the ideal complete spectrum determines every smooth admissible odd inverse-speed profile. The extracted lower profile is exponentially smoothing; on smooth Sobolev-prior classes we derive the sharp conditional logarithmic inverse modulus, including its leading constant and geometric admissibility margins. This is a stability statement for ideal extracted profile data, not recovery from noisy finite spectra. All arguments are analytic.

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1 The model and the simultaneous conclusion

The leading asymptotic term of a spectrum need not retain every geometric parameter. In the fixed tube below, it erases all odd logarithmic Fourier modes of the inverse twisting speed. We show that the next heat coefficient retains every one of them. This is an identifiability theorem for an ideal complete spectrum: it is not a claim of stable inversion from noisy finite data. Indeed the same multiplier estimates show exponential smoothing and failure of a local Hölder inverse for the extracted lower profile.

The mechanism has a broader form: Theorem 7.1 proves complete monotonicity of the full relative heat trace for nested strip junctions with arbitrary widths and offsets. It does not assert new tube multiplier noncancellation for those geometries. For the present model, Theorem 10.1 gives the optimal logarithmic recovery scale from the ideal lower profile under a Sobolev prior. Spectral cutoff and single-frequency lower tests are classical inverse-problem tools [8, 9]; the physical profile normalization and admissibility conditions are made explicit here.

Fix the connected polar section

$$\omega = \operatorname{int}\left(\overline{(1,3)\times(0,1)} \cup \overline{(3,\sqrt{11})\times(0,2)}\right). \tag{1.1}$$

The aperture $r = 3$, $0 < \vartheta < 1$, is interior. A tube end is parametrized by

$$(a, r, \vartheta) \longmapsto (a, r\cos(\vartheta + \Theta(a)), r\sin(\vartheta + \Theta(a))), \qquad (r, \vartheta) \in \omega.$$

The positive speed $b = \Theta'$ is smooth and strictly increasing on a remote end. Its inverse, in a fixed absolute axial coordinate, is

$$R(y) = yh(\log y), \qquad \tau = \frac{\pi}{\log 2}, \tag{1.2}$$

for sufficiently large y. Here h is a real smooth $2\log 2$-periodic function with $h > 0$ and $h + h' > 0$. A bounded smooth core uses the same section, and a one-end model has a Dirichlet cap at its fixed axial origin. We also consider the specified two-end model with identical tails. These restrictions are part of the inverse problem.

Let H_h be the Euclidean Dirichlet Laplacian. The actual-tube heat theorem proved in Section 4 gives

$$\operatorname{Tr} e^{-H_h/T^2} = T^4 A_h(\log T) + T^3 \left\{ c_{\mathrm{core}} + \sum_{k\in\mathbb{Z}} h_k C(z_k) e^{ik\tau \log T} \right\} + o(T^3), \quad z_k = 1 + ik\tau, \tag{1.3}$$

where $h(u) = \sum h_k e^{ik\tau u}$. A change of bounded core-volume normalization affects only c_{core}. The two profiles double for the specified identical two ends. For smooth h both Fourier profiles converge absolutely, as verified below. We recall the reduction and its analytic normalization in Section 2. Explicitly, the leading profile is

$$A_h(u) = \sum_k h_k \frac{z_k}{4\pi}(4 + 2^{z_k+1}) M_1(z_k) e^{ik\tau u}, \tag{1.4}$$

where M_1 is defined below. Thus the leading subtraction used for recovery is specified independently of the unknown odd coefficients.

The leading multiplier vanishes at every odd k. In particular, for $h(u) = 1 + \epsilon\cos(k\tau u)$ with odd k, $A_h = 1/(6\pi)$, independent of ϵ. The conclusion is leading-profile blindness.

Theorem 1.1 (All original odd modes)**.** *For every odd integer k, put $s_k = (z_k + 1)/2$. The coefficient in* (1.3) *satisfies the fully analytic strict bound*

$$\frac{|z_k\Gamma(s_k)|}{160\sqrt{\pi}} < |C(z_k)| < \frac{|z_k\Gamma(s_k)|}{2\sqrt{\pi}}. \tag{1.5}$$

Consequently the ideal complete eigenvalue multiset determines the amplitude in every fixed pure family

$$R_\epsilon(y) = y\{1 + \epsilon\cos(k\tau\log y)\}, \qquad |\epsilon| < (1 + k^2\tau^2)^{-1/2}.$$

The lower heat profile is also injective on all smooth admissible odd Fourier profiles $h = 1 + \sum_{k \text{ odd}} h_k e^{ik\tau u}$, including infinite series, with conjugate-symmetric coefficients and $h > 0$, $h + h' > 0$. These conclusions hold for one specified end or identical specified two ends.

The main assertion is simultaneous noncancellation on the full odd lattice. The proof first derives the actual Euclidean heat expansion, including the connected junction contribution. It then constructs a positive measure for the *full* connected relative heat trace D. The corner remainder $E = D - Z_1/2$ is not assigned a positive spectral measure.

Exploding-twist confinement and spectral estimates have classical foundations [5, 4]. Large-time heat estimates for locally twisted tubes [6] concern a different regime. Here the small-time, high-energy coefficient retains information lost by the leading width moment. Mellin methods for logarithmic periodicity are standard [7]; the issue is noncancellation of the exact connected coefficient.

2 The connected step and the actual heat reduction

Write

$$\mathcal{S} = \operatorname{int}\left(\overline{(-\infty,0)\times(0,1)} \cup \overline{(0,\infty)\times(0,2)}\right), \qquad Z_a(t) = \sum_{n\geq 1} e^{-t(\pi n/a)^2}.$$

The Dirichlet operator L is compared with L_{cut}, which imposes an additional Dirichlet condition at $\{0\}\times(0,1)$. Define $D(t)$ as the increasing finite-window limit of the connected-minus-cut heat trace, with Dirichlet caps at $X = \pm\ell$, as $\ell \to \infty$. This subtracts no separately infinite traces.

The killed Brownian-bridge diagonal formula identifies the difference with surviving loops that hit the aperture. Enlarging the step to the full width-two strip gives

$$\tfrac{1}{2}Z_1(t) \leq D(t) \leq \tfrac{1}{2}Z_2(t). \tag{2.1}$$

Indeed a horizontal bridge of generator ∂_X^2 based at X hits zero with probability $e^{-X^2/t}$, whose free diagonal integral over $\mathbb{R}$ is $1/2$. Restricting instead to the width-one strip proves the lower bound. These arguments prove finiteness, continuity for $t > 0$, integrability at zero, and exponential decay at infinity.

Here D_a denotes the finite-window difference with caps at $X = \pm a$. For completeness the cap error is bounded by

$$0 \leq D(t) - D_a(t) \leq Z_2(t)\left[\frac{a}{\sqrt{\pi t}}e^{-a^2/(4t)} + \tfrac{1}{2}\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)\right]. \tag{2.2}$$

For $|X| \leq a/2$, use the union bound for hitting either cap; its integral is the first term. Outside this interval use the aperture-hit Gaussian, giving the second term. The vertical domain may be enlarged to width two in both estimates.

Put $E = D - Z_1/2$. Translate the reflex vertex to the origin. The step is the wedge $\{Y < 0\} \cup \{X > 0\}$ restricted by $-1 < Y < 1$. Then E counts surviving loops hitting both internal rays $\{X = 0, Y < 0\}$ and $\{Y = 0, X > 0\}$. Scaling time to one shrinks only the allowed horizontal strip, so E is nonnegative and nonincreasing. Its zero-time limit is the unconfined both-ray loop mass

$$c(3\pi/2) + c(\pi/2) = \tfrac{1}{36}, \qquad c(\alpha) = \frac{\pi^2 - \alpha^2}{24\pi\alpha}.$$

The polygon corner formula [1] applies first in a bounded window; the Gaussian bridge bound removes its distant caps. Thus

$$0 \leq E(t) \leq \tfrac{1}{36}, \qquad E(t) \leq \tfrac{1}{2}(Z_2(t) - Z_1(t)). \tag{2.3}$$

Define, for $\operatorname{Re} z > 0$,

$$M_1(z) = \int_0^\infty v^z Z_1(v^2)\,\mathrm{d}v = \frac{\Gamma((z+1)/2)\zeta(z+1)}{2\pi^{z+1}}, \qquad \mathcal{M}_E(z) = \tfrac{1}{2}\int_0^\infty t^{(z-1)/2}E(t)\,\mathrm{d}t.$$

All these integrals converge absolutely. The actual heat reduction is

$$C(z) = \frac{z}{2\sqrt{\pi}}\left\{3M_D(z) - M_1(z) - \frac{3+\sqrt{11}}{4}\,2^{z+1}M_1(z)\right\}, \quad M_D(z) = \int_0^\infty v^z D(v^2)\,\mathrm{d}v. \tag{2.4}$$

At odd k, $2^{z_k+1} = -4$, so, writing $A = 7/2 + \sqrt{11}$,

$$C(z_k) = \frac{z_k}{2\sqrt{\pi}}\{AM_1(z_k) + 3\mathcal{M}_E(z_k)\}. \tag{2.5}$$

$$\int_0^\infty v^z Z_a(v^2)\,\mathrm{d}v = a^{z+1}M_1(z), \qquad \operatorname{Re} z > 0. \tag{2.6}$$

This follows by termwise Gaussian integration. The next theorem and its proof identify the coefficient for the actual Euclidean operator.

Theorem 2.1 (Actual-tube heat expansion)**.** *For the model and admissibility assumptions of Section 1, the Dirichlet operator has compact resolvent and finite heat trace. Equation* (1.3) *holds with the displayed leading profile and coefficient* (2.4)*. The core-volume convention affects only the constant lower coefficient; identical specified ends add their profiles.*

3 A uniform integrated radial expansion

For $\eta > 0$, $v > 0$ and $\xi \in \mathbb{R}$ let

$$K_\eta(v,\xi) = -\eta^2\partial_r^2 - v^2\partial_q^2 + \frac{\xi^2}{v^2 r^2}$$

be the Dirichlet operator on the connected section (1.1), with measure $\mathrm{d}r\,\mathrm{d}q$. Put

$$J_\eta(v) = \frac{1}{2\pi}\int_{\mathbb{R}} \operatorname{Tr} e^{-K_\eta(v,\xi)}\,\mathrm{d}\xi, \quad G(v) = 3D(v^2) - Z_1(v^2) - \frac{3+\sqrt{11}}{4}Z_2(v^2).$$

Proposition 3.1. *Uniformly for $v > 0$ and sufficiently small $\eta > 0$,*

$$J_\eta(v) = \frac{v}{4\pi\eta}\left[Z_1(v^2)\int_1^3 r\,\mathrm{d}r + Z_2(v^2)\int_3^{\sqrt{11}} r\,\mathrm{d}r\right] + \frac{v}{2\sqrt{\pi}}G(v) + E_\eta(v), \tag{3.1}$$

$$|E_\eta(v)| \le C\eta v Z_2(v^2), \qquad \int_0^\infty |E_\eta(v)|\,\mathrm{d}v \le C\eta/3. \tag{3.2}$$

Proof. Integrate ξ in the killed Brownian bridge formula first. If ρ_u is the radial path, the exact Gaussian integration is

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-(\xi/v)^2\int_0^1 \rho_u^{-2}\,\mathrm{d}u}\,\mathrm{d}\xi = \frac{v}{2\sqrt{\pi}}F[\rho], \qquad F[\rho] = \left(\int_0^1 \rho_u^{-2}\,\mathrm{d}u\right)^{-1/2}.$$

On surviving paths $1 \le \rho \le \sqrt{11}$, so F is bounded and Lipschitz in the path supremum norm. Extend r^{-2} to a positive bounded C^2 function on the real line, agreeing near this interval. For an unrestricted centered bridge $\rho_u = r + \eta\beta_u$, Taylor expansion gives, uniformly in r in the interval,

$$F[r+\eta\beta] = r + \eta\int_0^1 \beta_u\,\mathrm{d}u + O(\eta^2 \sup|\beta|^2). \tag{3.3}$$

The first variation has mean zero. Multiplying by the radial free diagonal $(4\pi\eta^2)^{-1/2}$, the unrestricted radial bulk is $(2\sqrt{\pi}\eta)^{-1}\int r\,\mathrm{d}r + O(\eta)$.

First cut at $r = 3$. On an arm $(l,u)\times(0,a)$ the angular trace factors as $Z_a(v^2)$. Kill radial bridges hitting l or u. At the lower endpoint replace F by l. The reflected Gaussian gives

$$l\int_l^u (4\pi\eta^2)^{-1/2} e^{-(r-l)^2/\eta^2}\,\mathrm{d}r = l/4 + O(e^{-c/\eta^2}).$$

The replacement error is $O(\eta)$. In fact, setting $x = (r-l)/\eta$, its bound is a constant times

$$\eta\int_0^\infty \mathbb{E}[(x + \sup|\beta|)\mathbf{1}_{\{\min\beta \le -x\}}]\,\mathrm{d}x < \infty.$$

Gaussian bridge supremum tails make this bound uniform. The upper endpoint contributes $u/4$ with the same error; bridges hitting both endpoints are exponentially small. Each cut arm therefore has weighted radial trace

$$\frac{1}{2\sqrt{\pi}\eta}\int_l^u r\,\mathrm{d}r - \frac{l+u}{4} + O(\eta).$$

Reconnect the aperture. The difference counts surviving paths hitting $r = 3$, now with weight F. Freeze F to 3. Setting $x = (r-3)/\eta$, the weight error is bounded by $C\eta(|x| + \sup|\beta|)$ on the hit event. Enlarge the angular domain to width two. For a fixed bridge the set of x for which it hits $-x$ has length at most $2\sup|\beta|$. Its integrated weighted error is therefore $O(\eta Z_2(v^2))$. The unweighted reconnection, in $X = (r-3)/\eta$, differs from $D(v^2)$ by $O(\eta^{-1}e^{-c/\eta^2}Z_2(v^2))$ by (2.2) at radial time one. This is uniform in v: after the anisotropic dilation, cap distance divided by square root of time remains a constant times η^{-1}. Thus reconnection contributes $v[3D(v^2) + O(\eta Z_2(v^2))]/(2\sqrt{\pi})$. The cut endpoint factors are $(1+3)Z_1/4$ and $(3+\sqrt{11})Z_2/4$. This proves (3.1) and its error. Finally (2.6) gives $\int_0^\infty vZ_2(v^2)\,\mathrm{d}v = 1/3$. □

4 From frozen fibres to the actual tube

We give the global comparison in detail because an error larger than T^3 would not identify the coefficient. In this section s denotes the original axial coordinate a, and q the angular coordinate ϑ. Write $D_j = -i\partial_j$. The actual form in measure $r\,\mathrm{d}r\,\mathrm{d}q\,\mathrm{d}s$ is

$$\int \left(|\partial_r u|^2 + r^{-2}|\partial_q u|^2 + |\partial_s u - b(s)\partial_q u|^2\right) r\,\mathrm{d}r\,\mathrm{d}q\,\mathrm{d}s. \tag{4.1}$$

The derivative margin in (1.2) implies $0 < c \le b' \le C$, $b'' = O(b^{-1})$, with analogous finite-order derivative bounds as needed below.

Lemma 4.1 (Weighted real normal form)**.** *On a sufficiently remote end, the form* (4.1) *is unitarily equivalent to a form whose local comparison with*

$$k[u] = \int \left(|D_r u|^2 + B(z)^2|D_q u|^2 + \frac{|D_z u|^2}{B(z)^2 r^2}\right) \mathrm{d}r\,\mathrm{d}q\,\mathrm{d}z \tag{4.2}$$

has relative error at most $C/B(z)^2$, plus a bounded absolute potential. The connected radial/angular domain remains fixed.

Proof. Use the exact angular clock

$$x = \Theta^{-1}(\Theta(s) + q), \qquad B = b(x).$$

Then $\partial_s - b(s)\partial_q = -b(s)\partial_q$ in the new coordinates, while the old angular derivative becomes $\partial_q + B^{-1}\partial_x$, and the axial Jacobian is $B/b(s)$. Uniformly for bounded q,

$$b(s)^2 = B^2 - 2b'(x)q + O(B^{-2}), \qquad B/b(s) = 1 + O(B^{-2}).$$

Remove polar measure by multiplication by $r^{1/2}$ and conjugate the positive axial density. The resulting form differs by weighted relative $O(B^{-2})$ and a bounded absolute term from

$$\int \left(|D_r u|^2 + r^{-2}|D_q u + B^{-1}D_x u|^2 + B^2|D_q u|^2\right).$$

For example, logarithmic density derivatives in q are $O(B^{-2})$, in x they are $O(B^{-3})$; Young's inequality absorbs their cross terms in this weighted error. The polar potential $-1/(4r^2)$ is bounded. These comparisons use forms, so introduce no unaccounted boundary terms on subsequent Neumann caps.

Remove the remaining large mixed direction by a real small shear

$$z = x + qg(x, r), \qquad g = -\frac{1}{B(1 + B^2r^2)}.$$

Here $g = O(B^{-3})$, $g_x = O(B^{-4})$, $g_r = O(B^{-3})$. For $a = B^2 + r^{-2}$ the leading z, q cross coefficient is $ag + r^{-2}/B = 0$. The residual is $O(B^{-5})$; the z, z coefficient is the Schur complement $(1 + B^2r^2)^{-1}$ up to smaller errors. Replacing it by $(B^2r^2)^{-1}$ costs relative $O(B^{-2})$. The radial shear has coefficient $O(B^{-3})D_rD_z$, again relative $O(B^{-2})$ against $|D_r|^2 + B^{-2}|D_z|^2$. Also $B(x) - B(z) = O(B^{-3})$ and the shear Jacobian is $1 + O(B^{-4})$, with differentiated bounds of the same harmless type. For sufficiently remote points this is a diffeomorphism. Both changes preserve (r, q) and all physical Dirichlet faces, including the interior aperture. Conjugating the last density proves the lemma. □

Lemma 4.2 (Axial caps and weighted comparison)**.** *Fix a sufficiently large speed cutoff B_0. The actual heat trace on the remote end differs from the continuous frozen diagonal trace*

$$\int_{R(B_0)}^{\infty} \frac{\mathrm{d}z}{2\pi} \int_{\mathbb{R}} \mathrm{d}p \, \mathrm{Tr}_\omega \exp\left[-T^{-2}\left(D_r^2 + B(z)^2 D_q^2 + \frac{p^2}{B(z)^2 r^2}\right)\right] \tag{4.3}$$

by $O(T^3/B_0) + O(T^{5/2}\log(2 + T)) + O(T^2)$, with constants uniform in sufficiently large B_0, except for fixed-core terms.

Proof. Insert normal z caps *after* the global coordinate changes. Partition the remote axis into intervals of length $\ell = T^{-1/2}$. For the frozen diagonal operator the Dirichlet and Neumann caps separate into sine and cosine momenta $p_j = \pi j/\ell$. The transverse heat trace decreases with $p \geq 0$ by form order. Thus the sum-versus-integral error is bounded by its $p = 0$ term; the D/N difference is exactly that term. Radial Neumann relaxation into the two bands gives

$$\mathrm{Tr}_\omega \, e^{-T^{-2}(D_r^2 + b^2 D_q^2)} \leq CTZ_2(b^2/T^2) \leq C\frac{T^2}{b} e^{-cb^2/T^2}. \tag{4.4}$$

Relaxing $r^{-2} \geq 1/11$ and summing the axial Gaussian gives the interval trace and its first energy moment bounds

$$C\left(\ell T^3 + \frac{T^2}{b}\right) e^{-cb^2/T^2}. \tag{4.5}$$

For the moment use $xe^{-x} \leq Ce^{-x/2}$; harmless polynomials in b/T may be absorbed by decreasing c.

On the cell with frozen speed b_j, Lemma 4.1 and bounded B' give relative error $e_j \leq C(\ell/b_j + b_j^{-2})$. Min–max and the scalar mean-value bound for the exponential control the trace error by Ce_j times (4.5) at weaker heat time. Crucially, keep this weight cell by cell. Since $\mathrm{d}z$ is comparable to $\mathrm{d}b$, the volume errors sum to

$$CT^3\ell \int_{B_0}^{\infty} b^{-1} e^{-cb^2/T^2} \, \mathrm{d}b \leq CT^{5/2}\log(2 + T),$$
$$CT^3 \int_{B_0}^{\infty} b^{-2} e^{-cb^2/T^2} \, \mathrm{d}b \leq CT^3/B_0.$$

The zero-momentum cap errors sum to $CT^2\ell^{-1}\int_{B_0}^{\infty} b^{-1}e^{-cb^2/T^2}\,\mathrm{d}b = O(T^{5/2}\log(2 + T))$. The second term of (4.5) multiplied by e_j is smaller at fixed B_0. Replacing cell values by the

continuous frozen integral uses the same relative energy squeeze. A bounded potential costs $O(T^2)$ since its exponent is $O(T^{-2})$ and the total trace is $O(T^4)$. This proves the claim. The estimates also give finiteness of the remote heat trace for every $T > 0$ and hence compact resolvent. □

Proof of Theorem 2.1. In (4.3) set $b = Tv$, $p = T^2\xi$, $\eta = T^{-1}$. The radial proposition gives its bulk

$$\frac{T^2}{4\pi}\int_{B_0}^{\infty} y\,\mathrm{d}R(y)[4Z_1(y^2/T^2) + Z_2(y^2/T^2)] \tag{4.6}$$

and boundary

$$\frac{T^3}{2\sqrt{\pi}}\int_{B_0/T}^{\infty} vR'(Tv)G(v)\,\mathrm{d}v, \tag{4.7}$$

with integrated error $O(T^2)$, because R' is bounded and (3.2) is integrable.

The transformed cap at speed B_0 differs from the physical axial cap by $O(B_0^{-1})$. Its core-volume discrepancy therefore costs at most CT^3/B_0. For fixed B_0 the core is a bounded Lipschitz domain. Its heat trace has its usual volume term and an $O_{B_0}(T^2)$ remainder. The formal extension of $R(y) = yh(\log y)$ to $y > 0$ has $R(0) = 0$. Its missing measure has $\int_0^{B_0} \mathrm{d}R = R(B_0)$, exactly the fixed-core length in the stated origin convention. Poisson summation for Z_a shows that extending (4.6) to zero adds this same volume term, with only $O_{B_0}(T^2)$ difference. An actual axial offset instead adds the stated constant volume coefficient.

By (2.1), $vG(v)$ is bounded at zero and exponentially decreasing at infinity. Extending the boundary lower limit to zero costs $O_{B_0}(T^2)$. Insert the absolutely convergent smooth Fourier series of $R'(Tv) = h(\log T + \log v) + h'(\log T + \log v)$. Equation (2.6) gives the leading profile; inserting G and using (2.4) gives the lower profile in (1.3). All integrals with logarithmic derivatives converge absolutely; smooth Fourier coefficients and the uniform absolute integral bounds justify the interchanges. At fixed B_0, the normalized limsup of the remaining error is at most C/B_0 by Lemma 4.2. Let T tend to infinity first, and then B_0 to infinity. This proves (1.3). The opposite twist sign has the same argument with the signed clock; its diagonal form depends on the speed magnitude. Joining two ends across a fixed cap costs $O(T^2)$ and gives the stated addition. □

5 The actual aperture trace

On $\mathcal{H} = L^2(0,1)$ let A_D and A_N be the positive DD and DN interval Laplacians, with bases

$$e_n = \sqrt{2}\sin(n\pi\vartheta), \qquad f_m = \sqrt{2}\sin((m-\tfrac{1}{2})\pi\vartheta).$$

Set $P(x) = (A_D + x)^{1/2}$, $R(x) = (A_N + x)^{1/2}$ and

$$\mathfrak{M}(x) = \tfrac{1}{2}(3P(x) + R(x)), \qquad f(x) = \mathrm{Tr}\{\mathfrak{M}(x)^{-1}\mathfrak{M}'(x)\}. \tag{5.1}$$

The aggregate form domain is $H_{00}^{1/2}(0,1)$, the DD zero-extension trace space. The DN root form has a larger natural domain and is restricted to this space.

Reflection about the midpoint of $(0,2)$ gives $J_0^* F(A_{(0,2),D})J_0 = (F(A_D) + F(A_N))/2$, where J_0 is zero extension. Thus the two half-strip Dirichlet-to-Neumann maps sum exactly to (5.1); they need not commute. Their Poisson extension K satisfies $K^*K = \mathfrak{M}'$ by direct exponential integration. The aperture flux matching equation gives

$$(L + x)^{-1} - (L_{\mathrm{cut}} + x)^{-1} = K\mathfrak{M}(x)^{-1}K^*.$$

This follows first for smooth aperture traces supported away from the endpoints, then by closure and coercivity of the stated form domain. The ideal estimates below show the correction is trace class. Taking the nonnegative heat-kernel diagonal integral and using Tonelli yields

$$f(x) = \int_0^\infty e^{-xt} D(t)\,\mathrm{d}t, \qquad x \geq 0. \tag{5.2}$$

Such gluing trace identities are classical; see [2] for the Dirichlet-to-Neumann/spectral-shift framework. Positivity of a spectral shift alone does not prove the stronger measure property needed here.

6 A positive-root matrix lemma

Lemma 6.1. *Let C_j be positive semidefinite Hermitian $n \times n$ matrices, with $\sum C_j > 0$, and let $a_j \geq a_* > 0$. For a finite sum*

$$F(z) = \sum_j C_j \sqrt{z + a_j}, \qquad g(z) = \operatorname{Tr}(F(z)^{-1} F'(z)),$$

using principal roots, g is the Stieltjes transform of a positive measure supported in $[a_*, \infty)$.

Proof. Off the negative cut the roots have positive real part, so F is invertible. On the upper cut bank, away from thresholds, write $F(-u + i0) = U + iV$, where

$$U = \sum_{a_j > u} C_j \sqrt{a_j - u}, \qquad V = \sum_{a_j < u} C_j \sqrt{u - a_j}.$$

Both are positive semidefinite, $U + V > 0$, $\dot{U} \leq 0$ and $\dot{V} \geq 0$. No commutation is used in

$$\operatorname{Re} F^{-1} = F^{-*} U F^{-1} \geq 0, \qquad \operatorname{Im} F^{-1} = -F^{-*} V F^{-1} \leq 0.$$

Finite trace cyclicity therefore proves

$$\frac{\mathrm{d}}{\mathrm{d}u} \arg\det F(-u + i0) = \operatorname{Tr}((\operatorname{Im} F^{-1})\dot{U}) + \operatorname{Tr}((\operatorname{Re} F^{-1})\dot{V}) \geq 0.$$

Since $\partial_z = -\partial_u$ on the bank, $\operatorname{Im} g(-u + i0) \leq 0$.

At a threshold a, use the local coordinate $w = \sqrt{z + a}$ on each chosen bank. Other roots are analytic locally in w; hence $\det F = w^m H(w)$ with $m \geq 0$ and $H(0) \neq 0$. The determinant is not identically zero by off-cut invertibility. It follows that

$$g(z) = \frac{m}{2(z + a)} + O(|z + a|^{-1/2}).$$

Thus any threshold atom has nonnegative mass $m/2$, and the residual small contour arc tends to zero. At infinity expansion of the finite root sum gives $g(z) = n/(2z) + O(z^{-2})$, uniformly outside a disk containing the thresholds. A keyhole Cauchy contour, with these small arcs and the decaying outer arc, proves

$$g(z) = \sum_a \frac{m_a/2}{z + a} + \int_{a_*}^\infty \frac{\rho(u)}{z + u}\,\mathrm{d}u, \qquad \rho(u) = -\pi^{-1} \operatorname{Im} g(-u + i0) \geq 0.$$

The density vanishes beyond the last threshold because there F and F' have a common scalar imaginary factor and g has real bank values. The remaining singularities are integrable inverse roots. This proves the claimed representation. □

7 Complete monotonicity for nested strip junctions

The positive-root argument is not restricted to the width ratio and endpoint alignment used above. The following result concerns the relative junction trace itself; no complete-monotonicity assertion is made for a remainder obtained by subtracting a narrow-strip contribution.

Theorem 7.1. *Let $I \subset J$ be nonempty bounded open intervals, allowing shared endpoints and $I = J$. Let A be the Dirichlet Laplacian on*

$$\Omega_{I,J} = \operatorname{int}\Big(\overline{(-\infty, 0) \times I} \cup \overline{(0, \infty) \times J}\Big),$$

and let B be obtained by inserting a Dirichlet cut at $\{0\} \times I$. Write A_K for the one-dimensional Dirichlet Laplacian on an interval K. With Dirichlet caps at longitudinal coordinates $\pm L$, define A_L, B_L. Then

$$D_{I,J}(t) = \lim_{L\to\infty} \Big(\operatorname{Tr} e^{-tA_L} - \operatorname{Tr} e^{-tB_L}\Big)$$

exists for every $t > 0$, and there is a positive Borel measure $\nu_{I,J}$ supported on $[\lambda_1(J), \infty)$ such that

$$D_{I,J}(t) = \int e^{-tu}\, \mathrm{d}\nu_{I,J}(u), \qquad \int (1+u)^{-1}\, \mathrm{d}\nu_{I,J}(u) < \infty. \tag{7.1}$$

In particular $D_{I,J}$ is completely monotone. Moreover,

$$\frac{1}{2} \operatorname{Tr} e^{-tA_I} \le D_{I,J}(t) \le \frac{1}{2} \operatorname{Tr} e^{-tA_J}. \tag{7.2}$$

For $I = J$, the lower bound is an equality.

Proof. The connected-minus-cut diagonal heat kernel counts Brownian bridges surviving in $\Omega_{I,J}$ and hitting the aperture. The finite-cap contributions are nonnegative and increase with L. Restricting these paths to the full narrow strip, or enlarging to the full wide strip and allowing any hit of the line $X = 0$, gives (7.2), because

$$\int_{\mathbb{R}} (4\pi t)^{-1/2} e^{-X^2/t}\, \mathrm{d}X = \frac{1}{2}.$$

This proves existence, finiteness and the bounds $D_{I,J}(t) = O(t^{-1/2})$ at zero and exponential decay at infinity. Bridge-kernel domination on compact positive-time intervals also gives continuity. No difference of infinite heat traces is used.

Let $E : L^2(I) \to L^2(J)$ be extension by zero and put

$$P_x = (A_I + x)^{1/2}, \quad Q_x = (A_J + x)^{1/2}, \quad M(x) = P_x + E^* Q_x E, \qquad x \ge 0.$$

The last equality is a form sum on $H^{1/2}_{00}(I)$, the form domain of P_0. This domain and its coercivity can be checked directly. The form of P_x is the minimum of

$$\int_{(0,\infty)\times I} (|\partial_s U|^2 + |\partial_y U|^2 + x|U|^2)\, \mathrm{d}s\, \mathrm{d}y$$

over halfstrip extensions of an aperture datum g, with zero lateral trace. The minimizing extension is $e^{-sP_x} g$, as follows from the sine basis. Extending such a function by zero to the wide halfstrip preserves its energy and is admissible there. Therefore

$$0 \le E^* Q_x E \le P_x, \qquad P_x \le M(x) \le 2P_x. \tag{7.3}$$

The narrow summand imposes exactly the stated domain; the second summand is bounded on that domain. For $S = P_0$ the normalized operator

$$B_x = S^{-1/2} M(x) S^{-1/2}$$

is bounded and satisfies $B_x \geq I$.

The Poisson operator K_x assigns to g its two decaying halfstrip extensions with traces g, Eg. Integrating the squared semigroups gives

$$K_x^* K_x = M'(x) = \frac{1}{2}\Big(P_x^{-1} + E^* Q_x^{-1} E\Big).$$

The halfstrip Green identity and the equality of the two aperture traces, with vanishing sum of outgoing normal derivatives, give

$$(A+x)^{-1} - (B+x)^{-1} = K_x M(x)^{-1} K_x^*.$$

One may first solve this jump equation for smooth aperture data and then pass to the closed forms. Since the interval eigenvalues grow quadratically, $S^{-1/2} \in \mathfrak{S}_4$ and $M'(x) \in \mathfrak{S}_2$. Consequently

$$T_x = S^{-1/2} M'(x) S^{-1/2} \in \mathfrak{S}_1.$$

Cyclicity for these factorizations yields

$$f(x) := \operatorname{Tr}(B_x^{-1} T_x) = \operatorname{Tr}\Big((A+x)^{-1} - (B+x)^{-1}\Big) = \int_0^\infty e^{-xt} D_{I,J}(t)\, \mathrm{d}t. \tag{7.4}$$

For the last identity, integrate the positive finite-cap bridge kernels and pass to the increasing limit; the theta bounds ensure integrability. Equivalently, use the diagonal kernel of the positive resolvent correction. In particular $f(0) < \infty$.

We now justify the positive-measure representation with both aperture limits. Let Π_N project onto the first N Dirichlet sine functions e_i on I. Compress M to this space and truncate its wide-root summand to the first L sine functions f_m on J. The resulting matrix is a finite sum of positive semidefinite matrix coefficients times $\sqrt{x+\alpha}$, with all $\alpha \geq \lambda_1(J)$; the narrow-root summand alone is positive definite. Lemma 6.1 therefore makes its logarithmic derivative a positive Stieltjes transform.

For fixed i, two integrations by parts give $|\langle E e_i, f_m \rangle| = O_i(m^{-2})$. The first boundary term vanishes because e_i vanishes at both ends of I; the second and the remaining integral have the stated bound. This also holds for shared endpoints and arbitrary offset of I in J. Thus the wide-root matrix-entry series and its derivative converge, with summable tails respectively $O(m^{-3})$ and $O(m^{-5})$. The narrow-root term gives uniform inverse control. Sending L to infinity proves that

$$f_N(x) = \operatorname{Tr}\Big(M_N(x)^{-1} M_N'(x)\Big)$$

is a positive Stieltjes transform, possibly initially with a nonnegative constant term. Here and below the measure support remains bounded below by $\lambda_1(J)$.

For clarity, the closure statement follows by compactifying positive measures with weight $(1+u)^{-1}$: boundedness of the transforms at $x = 1$ bounds the weighted masses. Finite support points give the limiting Stieltjes measure; mass at infinity gives only a nonnegative constant. In the present finite-N limit, $M_N(x) \geq \sqrt{x} I$ and $M_N'(x) \leq x^{-1/2} I$, so $f_N(x) \leq N/x$ and that constant is zero.

For the second limit work in the S-normalized coordinates. Coercivity $\Pi_N B_x \Pi_N \geq I$ and density of the sine spaces imply that the inverse Galerkin operators, extended by zero, converge strongly to B_x^{-1}. Indeed the variational Galerkin equation bounds the approximation error by $\|B_x\|$ times the best approximation error. Also $\Pi_N T_x \Pi_N \to T_x$ in trace norm. Hence $f_N(x) \to f(x)$. The same positive-measure compactification gives a Stieltjes measure for f, with at most an additional nonnegative constant. This constant again vanishes: the explicit eigenvalue series

give $M'(x) \to 0$ in $\mathfrak{S}_2$ as $x \to \infty$, by dominated convergence, whereas $\|B_x^{-1}\| \le 1$. Schatten Hölder with exponents $(4, 2, 4)$ therefore gives $\|T_x\|_1 \to 0$ and $f(x) \to 0$. We have proved

$$f(x) = \int_{[\lambda_1(J),\infty)} \frac{\mathrm{d}\nu_{I,J}(u)}{x+u}.$$

Evaluation at $x = 1$ gives the weighted-mass assertion.

The positive function $\int e^{-tu}\,\mathrm{d}\nu_{I,J}(u)$ has Laplace transform $f(x)$ by Tonelli. Weighted finite mass makes it locally integrable at zero. Uniqueness of Laplace transforms in (7.4), and continuity at every positive time, identify it with $D_{I,J}$. Finally differentiation is justified by $u^m e^{-tu} \le C_{m,t}(1+u)^{-1}$, proving complete monotonicity. When $I = J$, $M = 2P_x$ gives $f(x) = \frac{1}{2}\operatorname{Tr}(A_I + x)^{-1}$ and the claimed equality. □

Remark 7.2. The theorem requires straight halfstrips and an aperture equal to the entire narrow interval. It does not address a finite intervening chamber, a nonnested aperture, or higher transverse dimension. In higher dimension the first relative resolvent need not be trace class, so the displayed trace argument would require a separate regularization. Nor does complete monotonicity by itself prove noncancellation of Mellin multipliers for a different tube geometry.

Theorem 7.3 (The fixed step measure)**.** *There is a positive measure ν, supported in $[\pi^2/4, \infty)$, such that*

$$f(x) = \int \frac{\mathrm{d}\nu(u)}{x+u}, \qquad D(t) = \int e^{-ut}\,\mathrm{d}\nu(u). \tag{7.5}$$

In particular D is completely monotone.

Proof. Apply Theorem 7.1 to $I = (0, 1)$ and $J = (0, 2)$. Its relative heat trace is exactly the connected step invariant D defined above. The aperture representation and (5.2) identify its resolvent trace with the present $f(x)$. The threshold is $\lambda_1(J) = \pi^2/4$. In this special case the sine overlaps are

$$\langle e_n, f_m \rangle = \frac{(-1)^{n+m} 2n}{\pi(n^2 - (m - \frac{1}{2})^2)},$$

consistent with the summable compression tails in the general proof. □

The Stieltjes framework and its closure principles are classical [3]. The bank-phase sign, exact aggregate domain and the two limits identify the measure for these noncommuting aperture operators. This theorem concerns D, not the difference $E = D - Z_1/2$.

8 An analytic aperture mass bound

Lemma 8.1. *With $m = \int_0^\infty E(t)\,\mathrm{d}t$ and $q_0 = \int_0^\infty tE(t)\,\mathrm{d}t$, one has*

$$q_0 < \tfrac{1}{81}, \qquad m < \frac{1}{27\sqrt{2}} < \frac{21}{800}, \qquad f(0) < \frac{1}{12} + \frac{21}{800}. \tag{8.1}$$

Proof. At $x = 0$ abbreviate

$$P = A_D^{1/2}, \quad R = A_N^{1/2}, \quad C = \{(3P+R)/2\}^{-1}, \quad A_0 = (3P^{-1}+R^{-1})/4, \quad B = (3P^{-3}+R^{-3})/8.$$

Differentiation of (5.2) and (5.1) gives

$$q_0 = \operatorname{Tr}(CB) + \operatorname{Tr}(CA_0CA_0) - a/2, \quad a = \operatorname{Tr} P^{-4} = \tfrac{1}{90}, \quad b = \operatorname{Tr} R^{-4} = 15a. \tag{8.2}$$

Indeed $\mathfrak{M}' = A_0$, $\mathfrak{M}'' = -B$, and $-f'(0) = \operatorname{Tr}(CB) + \operatorname{Tr}(CA_0CA_0)$; the derivative of the $Z_1/2$ part contributes $a/2$. Difference quotients are justified in the stated ideals: $\mathfrak{M}'$ is Hilbert–Schmidt and $\mathfrak{M}''$ trace class locally on the positive axis, while C and its derivative are bounded. Exponential heat decay also permits the derivative at zero.

The order $C \le (2/3)P^{-1}$ and Schatten Hölder give

$$\operatorname{Tr}(CB) \le \frac{3a + a^{1/4}b^{3/4}}{12} < \frac{11a}{12},$$

using $15^{3/4} < 8$. For the quadratic term, compare the positive compact operators $A_0^{1/2}CA_0^{1/2}$ and $(2/3)A_0^{1/2}P^{-1}A_0^{1/2}$. Eigenvalue monotonicity bounds the squared traces; no operator monotonicity of the square is used. Expansion yields

$$\begin{aligned}\operatorname{Tr}(CA_0CA_0) &\le \frac{9a + 6\operatorname{Tr}(P^{-3}R^{-1}) + \operatorname{Tr}(P^{-1}R^{-1}P^{-1}R^{-1})}{36}\\ &\le \frac{9a + 6a^{3/4}b^{1/4} + \sqrt{ab}}{36} < \frac{25a}{36}.\end{aligned}$$

Here $15^{1/4} < 2$, $\sqrt{15} < 4$, and the noncommuting last term is bounded by Hilbert–Schmidt Cauchy–Schwarz:

$$\operatorname{Tr}(P^{-1}R^{-1}P^{-1}R^{-1}) \le \operatorname{Tr}(P^{-2}R^{-2}) \le \sqrt{ab}.$$

Substituting in (8.2) gives $q_0 < (11/12 + 25/36 - 1/2)a = 1/81$.

The height constraint $0 \le E \le H = 1/36$ implies for any $T > 0$

$$m - \frac{q_0}{T} = \int_0^\infty (1 - t/T)E(t)\,\mathrm{d}t \le \frac{HT}{2}.$$

Optimizing gives $m^2 \le 2Hq_0$, whence $m < 1/(27\sqrt{2}) < 21/800$. Finally $\int Z_1/2 = 1/12$, proving the last bound. □

9 Uniform Mellin domination and spectral recovery

By Theorem 7.3 and the mass lemma the Mellin function

$$\mathcal{Z}_D(s) = \int u^{-s}\,\mathrm{d}\nu(u), \qquad \operatorname{Re} s = 1,$$

is absolutely convergent, and

$$|\mathcal{Z}_D(1 + i\beta)| \le f(0). \tag{9.1}$$

The Gamma integral and absolute Fubini identify it with

$$\mathcal{Z}_D(s) = \Gamma(s)^{-1}\int_0^\infty t^{s-1}D(t)\,\mathrm{d}t = \tfrac{1}{2}\mathcal{Z}_1(s) + \frac{2\mathcal{M}_E(2s-1)}{\Gamma(s)}, \quad \mathcal{Z}_1(s) = \pi^{-2s}\zeta(2s).$$

For every real β, the absolutely convergent Euler product gives

$$|\zeta(2 + 2i\beta)| \ge \prod_p (1 + p^{-2})^{-1} = \frac{\zeta(4)}{\zeta(2)} = \frac{\pi^2}{15}, \qquad |\mathcal{Z}_1(1 + i\beta)| \ge \tfrac{1}{15}. \tag{9.2}$$

Substitution in (2.5) gives

$$C(z_k) = \frac{3z_k\Gamma(s_k)}{4\sqrt{\pi}}\left\{\mathcal{Z}_D(s_k) + (A/3 - 1/2)\mathcal{Z}_1(s_k)\right\}.$$

For $\operatorname{Re} s = 1$, reverse triangle inequality, (8.1) and (9.2) show

$$|\mathcal{Z}_D(s) + (A/3 - 1/2)\mathcal{Z}_1(s)| \geq \frac{A/3 - 1/2}{15} - f(0)$$
$$> \frac{2 + 829/250}{45} - \frac{1}{12} - \frac{21}{800} = \frac{171}{20000} > \frac{1}{120},$$

where $(829/250)^2 < 11$. This proves (1.5) simultaneously for every positive odd k. Reality gives the negative frequencies by conjugation. Gamma has no zeros, so there is no residual finite gap and no cutoff beyond which modes need a different argument.

For the upper bound, the same positive measure gives $|M_D(z_k)| \leq |\Gamma(s_k)|f(0)/2$, while the absolutely convergent zeta series gives $|M_1(z_k)| \leq |\Gamma(s_k)|/12$. Since $3M_D + (2 + \sqrt{11})M_1 = AM_1 + 3\mathcal{M}_E$, we obtain

$$|C(z_k)| \leq \frac{|z_k\Gamma(s_k)|}{2\sqrt{\pi}} \left\{ \frac{3}{2} f(0) + \frac{2 + \sqrt{11}}{12} \right\} < \frac{|z_k\Gamma(s_k)|}{2\sqrt{\pi}}.$$

Indeed $f(0) < 1/12 + 21/800$ and $\sqrt{11} < 10/3$ make the braces strictly less than one. This proves both sides of (1.5).

For smooth periodic h, its Fourier coefficients decay faster than any power. The absolute Mellin integrals in (2.4), at $\operatorname{Re} z = 1$, give the crude bound $|C(z_k)| \leq C(1 + |k|)$ for all integers k; the leading multiplier has the same polynomial bound. Thus both profile series in (1.3) converge absolutely and uniformly. More directly, $R'(Tv) = h(\log T + \log v) + h'(\log T + \log v)$ has an absolutely convergent Fourier series, and its uniform sum may be integrated against the integrable heat kernel in Section 2. The inherited actual-tube remainder uses the bounds on this fixed smooth admissible profile, not finiteness of its Fourier support. Hence (1.3) applies to these infinite smooth profiles with the same normalization.

To finish the spectral statement, set $u = \log T$ and subtract the known leading term in (1.3). Its normalized remainder is a periodic lower profile plus $o(1)$. For nonzero integer j, the limit

$$\lim_{u_0 \to \infty} \frac{1}{2\log 2} \int_{u_0}^{u_0 + 2\log 2} e^{-ij\tau u} e^{-3u} \left\{ \operatorname{Tr} e^{-H_h e^{-2u}} - e^{4u} A_h(u) \right\} \mathrm{d}u = h_j C(z_j)$$

follows from periodic orthogonality and the asymptotic remainder. The constant core coefficient disappears. Dividing by the known nonzero multiplier recovers each odd h_j. For a pure cosine $h_k = h_{-k} = \epsilon/2$, so it determines ϵ. For two identical specified ends both quantities double. This proves Theorem 1.1 in its stated scope.

Corollary 9.1 (No local Hölder inverse from the lower profile)**.** *Let $\mathcal{B}_h(u) = \sum_{k \text{ odd}} h_k C(z_k) e^{ik\tau u}$ denote the oscillatory lower heat profile, with its constant core term removed. For every fixed nonnegative integer a, no constants $K, \gamma > 0$ give*

$$\|h - 1\|_{C^0} \leq K \|\mathcal{B}_h - \mathcal{B}_1\|_{C^0}^{\gamma}$$

on any C^a neighborhood of 1 among smooth admissible odd profiles. This remains false if h and $h + h'$ are required to exceed $1/2$.

Proof. For positive odd $k \to \infty$, put

$$\epsilon_k = (1 + k^2\tau^2)^{-(a+2)/2}, \qquad h^{(k)}(u) = 1 + \epsilon_k \cos(k\tau u).$$

Then $h^{(k)} \to 1$ in C^a, and $\epsilon_k\sqrt{1 + k^2\tau^2} \to 0$, ensuring the stated strict admissibility margins for all sufficiently large k. Conjugate symmetry and the two-sided multiplier estimate give

$$\|h^{(k)} - 1\|_{C^0} = \epsilon_k, \qquad \|\mathcal{B}_{h^{(k)}} - \mathcal{B}_1\|_{C^0} = \epsilon_k |C(z_k)| \leq C\epsilon_k k^{3/2} e^{-\pi\tau k/4}.$$

For the last estimate use the classical identity $|\Gamma(1 + i\beta)|^2 = \pi\beta/\sinh(\pi\beta)$ with $\beta = k\tau/2$. For any fixed $K, \gamma > 0$, the ratio of the left side of the claimed inverse estimate to its right side grows exponentially, since ϵ_k is polynomial in k. This contradicts the estimate in every indicated neighborhood. □

10 A sharp conditional logarithmic inverse modulus

Here the data are the ideal odd part of the lower periodic heat profile, not a finite list of eigenvalues. All norms on a period use the normalized measure $\mathrm{d}u/L$, where $L = 2\log 2$ and $\tau = 2\pi/L$. For a periodic function define

$$\|g\|_{H^s}^2 = \sum_{k\in\mathbb{Z}} (1+\tau^2 k^2)^s |g_k|^2.$$

Write $C_k = C(z_k)$ for the multiplier of the kth lower-profile harmonic. Let $\mathcal{K}$ denote the odd-profile multiplier operator $(\mathcal{K}g)_k = C_k g_k$ for odd k, with all even coefficients zero. The preceding multiplier bounds and $|\Gamma(1+ib)|^2 = \pi b/\sinh(\pi b)$ give constants $c, C > 0$ such that

$$ck^{3/2}e^{-\gamma k} \le |C_k| \le Ck^{3/2}e^{-\gamma k}, \qquad \gamma = \frac{\pi\tau}{4}, \tag{10.1}$$

for all sufficiently large positive odd k. Every remaining odd multiplier is nonzero. The exact phases play no role in the following L^2 statement.

Theorem 10.1. *Fix $s > 1$ and $M > 0$. Let $\mathcal{A}_{s,M}$ be the class of real smooth L-periodic functions $h = 1 + g$ such that*

$$g_k = 0 \quad (k \text{ even}), \qquad \|g\|_{H^s} \le M, \qquad h \ge \tfrac{1}{2}, \quad h + h' \ge \tfrac{1}{2}.$$

For $\delta > 0$ define the pairwise conditional modulus

$$\Omega_{s,M}(\delta) = \sup\left\{\|h_1 - h_2\|_2 : h_1, h_2 \in \mathcal{A}_{s,M}, \quad \|\mathcal{K}(h_1 - h_2)\|_2 \le \delta\right\}.$$

Then

$$\lim_{\delta\downarrow 0} [\log(M/\delta)]^s \Omega_{s,M}(\delta) = 2M\left(\frac{\pi}{4}\right)^s. \tag{10.2}$$

In particular the logarithmic power and the displayed leading constant are sharp on this class of admissible inverse-speed profiles.

Proof. Put $R = 2M$ and $\ell = \log(R/\delta)$. For $g = h_1 - h_2$ we have $\|g\|_{H^s} \le R$ and $\|\mathcal{K}g\|_2 \le \delta$. Choose a positive odd integer K within bounded distance of

$$\frac{\ell - (s+2)\log\ell}{\gamma}.$$

For all sufficiently large K, (10.1) and the finitely many nonzero initial multipliers imply

$$\min_{\substack{|k|\le K \\ k \text{ odd}}} |C_k| \ge c_1 K^{3/2} e^{-\gamma K}$$

with $c_1 > 0$. Indeed the envelope on the right eventually decreases to zero, below the positive minimum of any fixed initial set. Split g into frequencies $|k| \le K$ and $|k| > K$. Parseval gives

$$\|g_{\mathrm{low}}\|_2 \le c_1^{-1}\delta K^{-3/2}e^{\gamma K} = O(R\ell^{-s-7/2}), \qquad \|g_{\mathrm{high}}\|_2 \le R(1+\tau^2K^2)^{-s/2}.$$

Since $K/\ell \to 1/\gamma$, this proves

$$\limsup_{\delta\downarrow 0} \ell^s \Omega_{s,M}(\delta) \le R(\gamma/\tau)^s.$$

The upper estimate uses no pointwise admissibility or Sobolev embedding.

For the reverse inequality choose a positive odd k within bounded distance of

$$\frac{\ell + (s+2)\log \ell}{\gamma},$$

and write

$$\phi_k(u) = \sqrt{2}\cos(k\tau u), \qquad a_k = M(1+\tau^2 k^2)^{-s/2}, \qquad h_\pm = 1 \pm a_k \phi_k.$$

The function ϕ_k has normalized L^2 norm one. Thus $\|h_\pm - 1\|_{H^s} = M$, and

$$\|h_+ - h_-\|_2 = 2a_k, \qquad \|\mathcal{K}(h_+ - h_-)\|_2 = 2a_k|C_k|.$$

The upper multiplier bound yields

$$2a_k|C_k| \le C_2 R k^{3/2-s} e^{-\gamma k} \le C_3 \delta \ell^{-2s-1/2} \le \delta$$

for sufficiently small δ. Moreover,

$$a_k\Big(\|\phi_k\|_\infty + \|\phi_k'\|_\infty\Big) \le \sqrt{2}a_k(1+k\tau) \longrightarrow 0,$$

because $s > 1$. Consequently $h_\pm \ge 1/2$ and $h_\pm + h_\pm' \ge 1/2$ for sufficiently small δ. These are actual smooth admissible profiles, even when the whole H^s ball does not embed in C^1. As $k/\ell \to 1/\gamma$,

$$\liminf_{\delta \downarrow 0} \ell^s \Omega_{s,M}(\delta) \ge 2M(\gamma/\tau)^s.$$

Finally $\gamma/\tau = \pi/4$, and replacing $\log(2M/\delta)$ by $\log(M/\delta)$ changes no leading limit. This proves (10.2). □

Remark 10.2. The cutoff proof and single-frequency lower test are the classical mechanisms of severely ill-posed diagonal inversion[8, 9]. Here the sharp constant comes from the actual Gamma attenuation and the physical Sobolev frequency τk. The theorem concerns fixed s, M and ideal lower-profile data. It is not a minimax statement for unprocessed spectral data, a finite-spectrum reconstruction theorem, or an endpoint assertion at $s = 1$. A simultaneous finite-spectrum use would additionally require an all-mode L^2 profile-error budget with the stated geometric priors; fixed-mode error estimates alone do not supply that budget.

11 Scope and limitations

The Stieltjes theory, Brownian bridge identities, polygon coefficients, trace ideals and Euler product are established methods. The contribution is their application through the positive-root phase argument, controlled aperture limits and analytic mass bound to the entire original blind lattice of the fixed section. In particular positive spectral-shift representations such as [2] are not being treated as positivity of their energy derivative. The latter sign is supplied here by Lemma 6.1.

The Gamma factor decays exponentially with frequency. The two-sided estimate quantifies exponential smoothing. Corollary 9.1 concerns inversion of the extracted lower profile; it does not define a metric on full spectra or rule out estimates using other spectral data. The identifiability statement uses the ideal complete spectrum. No finite-prefix recovery or noise tolerance is asserted here. Such an inverse problem additionally requires controlled geometry, a complete spectral prefix, certified uncertainties and independent tail bounds.

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