Audibility of all odd blind modes in rapidly twisted polar-strip tubes

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Abstract

For a fixed connected polar section, the leading Weyl profile of a rapidly twisted Euclidean tube is independent of every odd logarithmic harmonic of inverse twisting speed. We prove that the next heat profile detects the entire odd lattice. A finite positive-root matrix lemma and two controlled aperture limits represent the full connected relative heat trace as the Laplace transform of a positive measure. An analytic mass bound and Euler-product domination then give a strict, frequency-dependent lower bound for every odd Mellin multiplier. Together with the established actual-tube heat expansion, this determines each admissible pure amplitude and every admissible finite odd Fourier profile from the complete spectrum. The argument is analytic; exponential decay of the bound records the conditioning of high modes.

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1 The model and the simultaneous conclusion

A geometric width moment can erase oscillations from the leading Weyl profile while a lower heat profile still detects them. We prove simultaneous detection of the entire original odd lattice on one fixed connected section. The argument combines a positive-measure representation of the connected aperture trace with analytic Mellin domination, producing a strict frequency-dependent lower bound.

Fix the connected polar section

$$\omega = \operatorname{int}\left(\overline{(1,3)\times(0,1)} \cup \overline{(3,\sqrt{11})\times(0,2)}\right). \tag{1.1}$$

The aperture $r = 3$, $0 < \vartheta < 1$, is interior. A tube end is parametrized by

$$(a, r, \vartheta) \longmapsto (a, r\cos(\vartheta + \Theta(a)), r\sin(\vartheta + \Theta(a))), \qquad (r, \vartheta) \in \omega.$$

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The positive speed $b = \Theta'$ is smooth and strictly increasing on a remote end. Its inverse, in a fixed absolute axial coordinate, is

$$R(y) = yh(\log y), \qquad \tau = \frac{\pi}{\log 2}, \tag{1.2}$$

for sufficiently large y. Here h is a real smooth $2\log 2$-periodic function with $h > 0$ and $h + h' > 0$. A bounded smooth core uses the same section, and a one-end model has a Dirichlet cap at its fixed axial origin. We also consider the specified two-end model with identical tails. These restrictions are part of the inverse problem.

Let H_h be the Euclidean Dirichlet Laplacian. The preceding actual-tube heat theorem [1, Theorem 1.1] gives

$$\operatorname{Tr} e^{-H_h/T^2} = T^4 A_h(\log T) + T^3 \left\{ c_{\mathrm{core}} + \sum_{k\in\mathbb{Z}} h_k C(z_k) e^{ik\tau \log T} \right\} + o(T^3), \quad z_k = 1 + ik\tau, \tag{1.3}$$

where $h(u) = \sum h_k e^{ik\tau u}$. A change of bounded core-volume normalization affects only c_{core}. The two profiles double for the specified identical two ends. For the finite Fourier profiles used below the sum is finite. We recall the reduction and its analytic normalization in Section 2. Explicitly, the known leading profile is

$$A_h(u) = \sum_k h_k \frac{z_k}{4\pi}(4 + 2^{z_k+1}) M_1(z_k) e^{ik\tau u},$$

where M_1 is defined below. Thus the leading subtraction used for recovery is specified independently of the unknown odd coefficients.

The leading multiplier vanishes at every odd k. In particular, for $h(u) = 1 + \epsilon\cos(k\tau u)$ with odd k, $A_h = 1/(6\pi)$, independent of ϵ. The conclusion is leading-profile blindness.

Theorem 1.1 (All original odd modes)**.** *For every odd integer k, put $s_k = (z_k + 1)/2$. The coefficient in* (1.3) *satisfies the fully analytic strict bound*

$$|C(z_k)| > \frac{|z_k \Gamma(s_k)|}{160\sqrt{\pi}} > 0. \tag{1.4}$$

Consequently the ideal complete eigenvalue multiset determines the amplitude in every fixed pure family

$$R_\epsilon(y) = y\{1 + \epsilon\cos(k\tau \log y)\}, \qquad |\epsilon| < (1 + k^2\tau^2)^{-1/2}.$$

The lower heat profile is also injective on the admissible finite odd Fourier profiles $h = 1 + \sum_{k \text{ odd}} h_k e^{ik\tau u}$, with conjugate-symmetric coefficients and $h > 0$, $h + h' > 0$. These conclusions hold for one specified end or identical specified two ends.

The main assertion is simultaneous noncancellation on the full odd lattice. Earlier work established the actual heat expansion and the first mode [1]; complex-resolvent certificates treated selected higher modes [2, 3]. They are quantitative predecessors, not prerequisites for the argument here. The new mechanism is a positive measure for the *full* connected relative heat trace D. The corner remainder $E = D - Z_1/2$ is not assigned a positive spectral measure.

2 The connected step and the actual heat reduction

Write

$$\mathcal{S} = \operatorname{int}\left(\overline{(-\infty, 0) \times (0, 1)} \cup \overline{(0, \infty) \times (0, 2)}\right), \qquad Z_a(t) = \sum_{n\geq 1} e^{-t(\pi n/a)^2}.$$

The Dirichlet operator L is compared with L_{cut}, which imposes an additional Dirichlet condition at $\{0\} \times (0,1)$. Define $D(t)$ as the increasing finite-window limit of the connected-minus-cut heat trace, with Dirichlet caps at $X = \pm\ell$, as $\ell \to \infty$. This subtracts no separately infinite traces.

The killed Brownian-bridge diagonal formula identifies the difference with surviving loops that hit the aperture. Enlarging the step to the full width-two strip gives

$$\tfrac{1}{2}Z_1(t) \le D(t) \le \tfrac{1}{2}Z_2(t). \tag{2.1}$$

Indeed a horizontal bridge of generator ∂_X^2 based at X hits zero with probability $e^{-X^2/t}$, whose free diagonal integral over $\mathbb{R}$ is $1/2$. Restricting instead to the width-one strip proves the lower bound. These arguments prove finiteness, continuity for $t > 0$, integrability at zero, and exponential decay at infinity.

Put $E = D - Z_1/2$. Translate the reflex vertex to the origin. The step is the wedge $\{Y < 0\} \cup \{X > 0\}$ restricted by $-1 < Y < 1$. Then E counts surviving loops hitting both internal rays $\{X = 0, Y < 0\}$ and $\{Y = 0, X > 0\}$. Scaling time to one shrinks only the allowed horizontal strip, so E is nonnegative and nonincreasing. Its zero-time limit is the unconfined both-ray loop mass

$$c(3\pi/2) + c(\pi/2) = \tfrac{1}{36}, \qquad c(\alpha) = \frac{\pi^2 - \alpha^2}{24\pi\alpha}.$$

The polygon corner formula [4] applies first in a bounded window; the Gaussian bridge bound removes its distant caps. Thus

$$0 \le E(t) \le \tfrac{1}{36}, \qquad E(t) \le \tfrac{1}{2}(Z_2(t) - Z_1(t)). \tag{2.2}$$

Define, for $\operatorname{Re} z > 0$,

$$M_1(z) = \int_0^\infty v^z Z_1(v^2)\,\mathrm{d}v = \frac{\Gamma((z+1)/2)\zeta(z+1)}{2\pi^{z+1}}, \qquad \mathcal{M}_E(z) = \tfrac{1}{2}\int_0^\infty t^{(z-1)/2}E(t)\,\mathrm{d}t.$$

All these integrals converge absolutely. The actual heat reduction is

$$C(z) = \frac{z}{2\sqrt{\pi}}\left\{3M_D(z) - M_1(z) - \frac{3+\sqrt{11}}{4}\,2^{z+1}M_1(z)\right\}, \quad M_D(z) = \int_0^\infty v^z D(v^2)\,\mathrm{d}v. \tag{2.3}$$

At odd k, $2^{z_k+1} = -4$, so, writing $A = 7/2 + \sqrt{11}$,

$$C(z_k) = \frac{z_k}{2\sqrt{\pi}}\{AM_1(z_k) + 3\mathcal{M}_E(z_k)\}. \tag{2.4}$$

For specificity we recall why the coefficient is an actual-tube invariant rather than a disconnected-width approximation. The uniform frozen radial theorem of [1] uses

$$K_\eta(v,\xi) = -\eta^2\partial_r^2 - v^2\partial_\vartheta^2 + \frac{\xi^2}{v^2r^2}, \qquad J_\eta(v) = \frac{1}{2\pi}\int_{\mathbb{R}} \operatorname{Tr} e^{-K_\eta(v,\xi)}\,\mathrm{d}\xi.$$

Its precise integrated expansion is

$$\begin{aligned} J_\eta(v) &= \frac{v}{4\pi\eta}\left\{4Z_1(v^2) + Z_2(v^2)\right\} + \frac{v}{2\sqrt{\pi}}G(v) + \mathcal{E}_\eta(v), \\ G(v) &= 3D(v^2) - Z_1(v^2) - \tfrac{1}{4}(3+\sqrt{11})Z_2(v^2), \\ |\mathcal{E}_\eta(v)| &\le C\eta v Z_2(v^2), \qquad \int_0^\infty v Z_2(v^2)\,\mathrm{d}v = \tfrac{1}{3}. \end{aligned} \tag{2.5}$$

The radial integrals are $\int_1^3 r\,\mathrm{d}r = 4$ and $\int_3^{\sqrt{11}} r\,\mathrm{d}r = 1$. To see the normalization directly, integrate ξ first in the killed bridge formula. Its path weight is $v(\int_0^1 \rho_s^{-2}\,\mathrm{d}s)^{-1/2}/(2\sqrt{\pi})$. For $\rho_s = r + \eta\beta_s$, this is $v/(2\sqrt{\pi})$ times $r + \eta\int \beta_s\,\mathrm{d}s + O(\eta^2 \sup|\beta|^2)$. The centered first variation vanishes in the unrestricted bulk. At an endpoint the integrated bridge-hit weight is $1/4$; replacing the path weight by its endpoint value has error $O(\eta)$. At the aperture the frozen weight is $r = 3$, and the true connected step gives $3D(v^2)$. The same Gaussian supremum estimate bounds its weight replacement by $O(\eta Z_2(v^2))$. Distant radial endpoints have error $O(\eta^{-1}e^{-c/\eta^2}Z_2(v^2))$. This proves the uniform error in (2.5), including $v \downarrow 0$.

For the global passage one uses the exact angular clock and a real shear, as in [1]. If $x = \Theta^{-1}(\Theta(a)+q)$ and $B = b(x)$, then the old vector field $\partial_a - b(a)\partial_\vartheta$ becomes $-b(a)\partial_q$ in the new variables, while the old angular derivative becomes $\partial_q + B^{-1}\partial_x$. The shear $z = x - q/[B(1+B^2r^2)]$ removes the principal mixed term. The resulting form is the diagonal model

$$\int \left\{|\partial_r u|^2 + B(z)^2|\partial_q u|^2 + (B(z)^2r^2)^{-1}|\partial_z u|^2\right\}\,\mathrm{d}r\,\mathrm{d}q\,\mathrm{d}z$$

up to local relative error C/B^2 and bounded scalar potentials. The bounds follow from b' bounded above and below and $b'' = O(1/b)$, consequences of (1.2) and strict admissibility. Above a fixed speed B_0, normal axial cells of length $T^{-1/2}$ give, by Dirichlet/Neumann bracketing and monotone momentum sums, error

$$O(T^{5/2}\log(2+T)) + O(T^3/B_0).$$

For example the relevant frozen cell trace and first energy moment are bounded by $C(\ell T^3 + T^2/b)e^{-cb^2/T^2}$, and the cap error is $CT^2b^{-1}e^{-cb^2/T^2}$. Summing the relative errors $C(\ell/b + b^{-2})$ gives the displayed estimate. Bounded potentials cost $O(T^2)$. The fixed core contributes its usual volume coefficient; its omitted inverse-speed bulk matches $\int_0^{B_0}\,\mathrm{d}R = R(B_0)$, up to the stated constant-volume convention.

Now put $\eta = T^{-1}$, $b = Tv$, and $p = T^2\xi$ in (2.5). The bounded density $R'(Tv) = h(\log T + \log v) + h'(\log T + \log v)$ and its uniform integrable remainder give the lower term

$$\frac{T^3}{2\sqrt{\pi}}\int_0^\infty vR'(Tv)G(v)\,\mathrm{d}v.$$

Fourier integration gives (2.3); the factor z is the derivative of y^z. The bulk has width factor $4+2^{z+1}$, which vanishes at odd k. For $h = 1$ its value is $T^4/(6\pi)$. First let $T \to \infty$, then $B_0 \to \infty$; the bound C/B_0 proves (1.3). This records the inherited heat lift and its normalization. The new proof below concerns its nonzero multipliers.

3 The actual aperture trace

On $\mathcal{H} = L^2(0,1)$ let A_D and A_N be the positive DD and DN interval Laplacians, with bases

$$e_n = \sqrt{2}\sin(n\pi\vartheta), \qquad f_m = \sqrt{2}\sin((m-\tfrac{1}{2})\pi\vartheta).$$

Set $P(x) = (A_D + x)^{1/2}$, $R(x) = (A_N + x)^{1/2}$ and

$$\mathfrak{M}(x) = \tfrac{1}{2}(3P(x) + R(x)), \qquad f(x) = \mathrm{Tr}\{\mathfrak{M}(x)^{-1}\mathfrak{M}'(x)\}. \tag{3.1}$$

The aggregate form domain is $H_{00}^{1/2}(0,1)$, the DD zero-extension trace space. The DN root form has a larger natural domain and is restricted to this space.

Reflection about the midpoint of $(0,2)$ gives $J_0^* F(A_{(0,2),D})J_0 = (F(A_D) + F(A_N))/2$, where J_0 is zero extension. Thus the two half-strip Dirichlet-to-Neumann maps sum exactly to (3.1);

they need not commute. Their Poisson extension K satisfies $K^*K = \mathfrak{M}'$ by direct exponential integration. The aperture flux matching equation gives

$$(L + x)^{-1} - (L_{\mathrm{cut}} + x)^{-1} = K\mathfrak{M}(x)^{-1}K^*.$$

This follows first for smooth aperture traces supported away from the endpoints, then by closure and coercivity of the stated form domain. The ideal estimates below show the correction is trace class. Taking the nonnegative heat-kernel diagonal integral and using Tonelli yields

$$f(x) = \int_0^\infty e^{-xt} D(t)\,\mathrm{d}t, \qquad x \geq 0. \tag{3.2}$$

Such gluing trace identities are classical; see [5] for the Dirichlet-to-Neumann/spectral-shift framework. Positivity of a spectral shift alone does not prove the stronger measure property needed here.

4 A positive-root matrix lemma

Lemma 4.1. *Let C_j be positive semidefinite Hermitian $n \times n$ matrices, with $\sum C_j > 0$, and let $a_j \geq a_* > 0$. For a finite sum*

$$F(z) = \sum_j C_j \sqrt{z + a_j}, \qquad g(z) = \mathrm{Tr}(F(z)^{-1}F'(z)),$$

using principal roots, g is the Stieltjes transform of a positive measure supported in $[a_*, \infty)$.

Proof. Off the negative cut the roots have positive real part, so F is invertible. On the upper cut bank, away from thresholds, write $F(-u + i0) = U + iV$, where

$$U = \sum_{a_j > u} C_j \sqrt{a_j - u}, \qquad V = \sum_{a_j < u} C_j \sqrt{u - a_j}.$$

Both are positive semidefinite, $U + V > 0$, $\dot{U} \leq 0$ and $\dot{V} \geq 0$. No commutation is used in

$$\mathrm{Re}\, F^{-1} = F^{-*}UF^{-1} \geq 0, \qquad \mathrm{Im}\, F^{-1} = -F^{-*}VF^{-1} \leq 0.$$

Finite trace cyclicity therefore proves

$$\frac{\mathrm{d}}{\mathrm{d}u} \arg\det F(-u + i0) = \mathrm{Tr}((\mathrm{Im}\, F^{-1})\dot{U}) + \mathrm{Tr}((\mathrm{Re}\, F^{-1})\dot{V}) \geq 0.$$

Since $\partial_z = -\partial_u$ on the bank, $\mathrm{Im}\, g(-u + i0) \leq 0$.

At a threshold a, use the local coordinate $w = \sqrt{z + a}$ on each chosen bank. Other roots are analytic locally in w; hence $\det F = w^m H(w)$ with $m \geq 0$ and $H(0) \neq 0$. The determinant is not identically zero by off-cut invertibility. It follows that

$$g(z) = \frac{m}{2(z + a)} + O(|z + a|^{-1/2}).$$

Thus any threshold atom has nonnegative mass $m/2$, and the residual small contour arc tends to zero. At infinity expansion of the finite root sum gives $g(z) = n/(2z) + O(z^{-2})$, uniformly outside a disk containing the thresholds. A keyhole Cauchy contour, with these small arcs and the decaying outer arc, proves

$$g(z) = \sum_a \frac{m_a/2}{z + a} + \int_{a_*}^\infty \frac{\rho(u)}{z + u}\,\mathrm{d}u, \qquad \rho(u) = -\pi^{-1}\,\mathrm{Im}\, g(-u + i0) \geq 0.$$

The density vanishes beyond the last threshold because there F and F' have a common scalar imaginary factor and g has real bank values. The remaining singularities are integrable inverse roots. This proves the claimed representation. □

5 Two limits and a positive measure for the full trace

Theorem 5.1. *There is a positive measure ν, supported in $[\pi^2/4, \infty)$, such that*

$$f(x) = \int \frac{\mathrm{d}\nu(u)}{x+u}, \qquad D(t) = \int e^{-ut}\,\mathrm{d}\nu(u). \tag{5.1}$$

In particular D is completely monotone.

Proof. Let Π_N project onto $e_1, \dots, e_N$. First truncate only the DN sum after compression, putting

$$\begin{aligned} M_{NL}(z) = {} & \tfrac{3}{2} \sum_{n \le N} e_n e_n^* \sqrt{z + \pi^2 n^2} \\ & + \tfrac{1}{2} \sum_{m \le L} (\Pi_N f_m)(\Pi_N f_m)^* \sqrt{z + \pi^2 (m - \tfrac{1}{2})^2}. \end{aligned}$$

The DD coefficient sum is $3I_N/2$. Lemma 4.1 applies to $\operatorname{Tr} M_{NL}^{-1} M_{NL}'$ with the same gap $\pi^2/4$, for every N, L. For fixed N and $x > 0$, the overlaps are $O_n(m^{-2})$; this follows also from

$$\langle e_n, f_m \rangle = \frac{(-1)^{n+m} 2n}{\pi(n^2 - (m - \frac{1}{2})^2)}.$$

Consequently the root-entry tails are $O(m^{-3})$, and their derivative tails are $O(m^{-5})$. The matrices and their derivatives converge in norm as $L \to \infty$. Their positive DD parts bound inverses uniformly, giving

$$\operatorname{Tr} M_{NL}^{-1} M_{NL}' \longrightarrow f_N(x) = \operatorname{Tr}\{(\Pi_N \mathfrak{M}(x) \Pi_N)^{-1} \Pi_N \mathfrak{M}'(x) \Pi_N\}.$$

We state the measure-closure detail explicitly. Weight each positive representing measure by $(1+u)^{-1}$; its mass equals the corresponding function value at 1. Bounded values give weak compactness on $[\pi^2/4, \infty]$. The kernels $(1+u)/(x+u)$ extend continuously to infinity with value 1. Thus a finite positive pointwise limit has a Stieltjes representation, possibly plus a nonnegative constant from mass escaping to infinity. The fixed gap is preserved. For fixed N, $M_N(x) \ge 3\sqrt{x} I_N/2$ and $M_N'(x) \le I_N/\sqrt{x}$, so $f_N(x) \le 2N/(3x)$. This removes that constant in the first limit.

For the second limit put $S = A_D^{1/2}$ and normalize the forms by

$$B_x = S^{-1/2} \mathfrak{M}(x) S^{-1/2}, \qquad T_x = S^{-1/2} \mathfrak{M}'(x) S^{-1/2}.$$

The form order $A_N \le A_D$ implies $R(x) \le P(x)$. Hence B_x is bounded and coercive, with $B_x \ge 3I/2$. Also T_x is positive trace class: $S^{-1/2} \in \mathfrak{S}_4$ and $\mathfrak{M}'(x) = (3P(x)^{-1} + R(x)^{-1})/4 \in \mathfrak{S}_2$, and the Schatten exponents $4, 2, 4$ give trace class. These estimates also justify the trace-class gluing formula used in (3.2). Since Π_N commutes with S, coercive bounded Galerkin convergence gives

$$(\Pi_N B_x \Pi_N)^{-1} \Pi_N \longrightarrow B_x^{-1} \quad \text{strongly}, \qquad \Pi_N T_x \Pi_N \longrightarrow T_x \quad \text{in trace norm}.$$

For the first assertion, approximate $B_x^{-1} v$ in the finite spans and use the Galerkin best approximation in its equivalent energy norm. Uniform boundedness and trace-norm convergence then imply

$$f_N(x) \longrightarrow \operatorname{Tr}(B_x^{-1} T_x) = f(x).$$

The equality follows from the closed-form inverse identity $\mathfrak{M}^{-1} = S^{-1/2} B_x^{-1} S^{-1/2}$ and the same trace ideals. Apply measure closure once more. By (3.2), integrability of D and dominated

convergence give $f(x) \to 0$ as $x \to \infty$, removing the possible constant. This proves the first identity in (5.1).

The weighted measure is finite. Its inverse Laplace kernel $G(t) = \int e^{-ut}\,\mathrm{d}\nu(u)$ is continuous for $t > 0$ and locally integrable, and Tonelli gives its Laplace transform $f(x)$. Uniqueness of Laplace transforms, followed by continuity, proves $G = D$. This is the second identity. Differentiating the exponentially convergent integrals for $t > 0$ proves complete monotonicity. □

The Stieltjes framework and its closure principles are classical [6]. The special bank-phase sign and the two limits above identify the measure for this particular noncommuting aperture. Theorem 5.1 is a statement about D, not about the difference $E = D - Z_1/2$.

6 An analytic aperture mass bound

Lemma 6.1. *With $m = \int_0^\infty E(t)\,\mathrm{d}t$ and $q_0 = \int_0^\infty tE(t)\,\mathrm{d}t$, one has*

$$q_0 < \tfrac{1}{81}, \qquad m < \frac{1}{27\sqrt{2}} < \frac{21}{800}, \qquad f(0) < \frac{1}{12} + \frac{21}{800}. \tag{6.1}$$

Proof. At $x = 0$ abbreviate

$$P = A_D^{1/2}, \quad R = A_N^{1/2}, \quad C = \{(3P+R)/2\}^{-1}, \quad A_0 = (3P^{-1}+R^{-1})/4, \quad B = (3P^{-3}+R^{-3})/8.$$

Differentiation of (3.2) and (3.1) gives

$$q_0 = \operatorname{Tr}(CB) + \operatorname{Tr}(CA_0CA_0) - a/2, \quad a = \operatorname{Tr} P^{-4} = \tfrac{1}{90}, \quad b = \operatorname{Tr} R^{-4} = 15a. \tag{6.2}$$

Indeed $\mathfrak{M}' = A_0$, $\mathfrak{M}'' = -B$, and $-f'(0) = \operatorname{Tr}(CB) + \operatorname{Tr}(CA_0CA_0)$; the derivative of the $Z_1/2$ part contributes $a/2$. Difference quotients are justified in the stated ideals: $\mathfrak{M}'$ is Hilbert–Schmidt and $\mathfrak{M}''$ trace class locally on the positive axis, while C and its derivative are bounded. Exponential heat decay also permits the derivative at zero.

The order $C \le (2/3)P^{-1}$ and Schatten Hölder give

$$\operatorname{Tr}(CB) \le \frac{3a + a^{1/4}b^{3/4}}{12} < \frac{11a}{12},$$

using $15^{3/4} < 8$. For the quadratic term, compare the positive compact operators $A_0^{1/2}CA_0^{1/2}$ and $(2/3)A_0^{1/2}P^{-1}A_0^{1/2}$. Eigenvalue monotonicity bounds the squared traces; no operator monotonicity of the square is used. Expansion yields

$$\begin{aligned}\operatorname{Tr}(CA_0CA_0) &\le \frac{9a + 6\operatorname{Tr}(P^{-3}R^{-1}) + \operatorname{Tr}(P^{-1}R^{-1}P^{-1}R^{-1})}{36} \\ &\le \frac{9a + 6a^{3/4}b^{1/4} + \sqrt{ab}}{36} < \frac{25a}{36}.\end{aligned}$$

Here $15^{1/4} < 2$, $\sqrt{15} < 4$, and the noncommuting last term is bounded by Hilbert–Schmidt Cauchy–Schwarz:

$$\operatorname{Tr}(P^{-1}R^{-1}P^{-1}R^{-1}) \le \operatorname{Tr}(P^{-2}R^{-2}) \le \sqrt{ab}.$$

Substituting in (6.2) gives $q_0 < (11/12 + 25/36 - 1/2)a = 1/81$.

The height constraint $0 \le E \le H = 1/36$ implies for any $T > 0$

$$m - \frac{q_0}{T} = \int_0^\infty (1 - t/T)E(t)\,\mathrm{d}t \le \frac{HT}{2}.$$

Optimizing gives $m^2 \le 2Hq_0$, whence $m < 1/(27\sqrt{2}) < 21/800$. Finally $\int Z_1/2 = 1/12$, proving the last bound. □

7 Uniform Mellin domination and spectral recovery

By Theorem 5.1 and the mass lemma the Mellin function

$$\mathcal{Z}_D(s) = \int u^{-s}\,\mathrm{d}\nu(u), \qquad \operatorname{Re} s = 1,$$

is absolutely convergent, and

$$|\mathcal{Z}_D(1+i\beta)| \le f(0). \tag{7.1}$$

The Gamma integral and absolute Fubini identify it with

$$\mathcal{Z}_D(s) = \Gamma(s)^{-1}\int_0^\infty t^{s-1}D(t)\,\mathrm{d}t = \tfrac{1}{2}\mathcal{Z}_1(s) + \frac{2\mathcal{M}_E(2s-1)}{\Gamma(s)}, \quad \mathcal{Z}_1(s) = \pi^{-2s}\zeta(2s).$$

For every real β, the absolutely convergent Euler product gives

$$|\zeta(2+2i\beta)| \ge \prod_p (1+p^{-2})^{-1} = \frac{\zeta(4)}{\zeta(2)} = \frac{\pi^2}{15}, \qquad |\mathcal{Z}_1(1+i\beta)| \ge \tfrac{1}{15}. \tag{7.2}$$

Substitution in (2.4) gives

$$C(z_k) = \frac{3z_k\Gamma(s_k)}{4\sqrt{\pi}}\left\{\mathcal{Z}_D(s_k) + (A/3-1/2)\mathcal{Z}_1(s_k)\right\}.$$

For $\operatorname{Re} s = 1$, reverse triangle inequality, (6.1) and (7.2) show

$$\begin{aligned}|\mathcal{Z}_D(s) + (A/3-1/2)\mathcal{Z}_1(s)| &\ge \frac{A/3-1/2}{15} - f(0)\\ &> \frac{2+829/250}{45} - \frac{1}{12} - \frac{21}{800} = \frac{171}{20000} > \frac{1}{120},\end{aligned}$$

where $(829/250)^2 < 11$. This proves (1.4) simultaneously for every positive odd k. Reality gives the negative frequencies by conjugation. Gamma has no zeros, so there is no residual finite gap and no cutoff beyond which modes need a different argument.

To finish the spectral statement, set $u = \log T$ and subtract the known leading term in (1.3). Its normalized remainder is a periodic lower profile plus $o(1)$. For nonzero integer j, the limit

$$\lim_{u_0\to\infty}\frac{1}{2\log 2}\int_{u_0}^{u_0+2\log 2} e^{-ij\tau u}e^{-3u}\left\{\operatorname{Tr} e^{-H_h e^{-2u}} - e^{4u}A_h(u)\right\}\mathrm{d}u = h_j C(z_j)$$

follows from periodic orthogonality and the asymptotic remainder. The constant core coefficient disappears. Dividing by the known nonzero multiplier recovers each odd h_j. For a pure cosine $h_k = h_{-k} = \epsilon/2$, so it determines ϵ. For two identical specified ends both quantities double. This proves Theorem 1.1 in its stated scope.

8 The ninth mode, reproducibility and limitations

The theorem is analytical and requires no numerical aperture data. For $k = 9$, put $\beta = 9\pi/(2\log 2)$. The elementary identity

$$|\Gamma(1+i\beta)|^2 = \frac{2\pi\beta}{e^{\pi\beta} - e^{-\pi\beta}}, \qquad |z_9|^2 = 1 + 4\beta^2,$$

together with (1.4), gives $|C(z_9)| > 1.5\cdot 10^{-14}$. The accompanying exact-rational scalar script checks this conservative decimal corollary using Machin bounds for π, the positive atanh series

for $\log 2$, and a Taylor series with geometric exponential tail. It also checks the rational comparisons in the mass and noncancellation estimates. This script certifies scalar inequalities; the operator theorem is proved above and needs no computed nodes.

The Stieltjes theory, Brownian bridge identities, polygon coefficients, trace ideals and Euler product are established methods. The contribution is their application through the positive-root phase argument, controlled aperture limits and analytic mass bound to the entire original blind lattice of the fixed section. In particular positive spectral-shift representations such as [5] are not being treated as positivity of their energy derivative. The latter sign is supplied here by Lemma 4.1.

The Gamma factor decays exponentially with frequency. The theorem therefore concerns identifiability from the complete spectrum, with frequency-dependent conditioning. The quantitative successor [7] gives finite-prefix recovery disks under controlled geometry, a complete spectral prefix, certified uncertainties and tail and multiplier bounds. Its geometric remainder constants remain analytic inputs; its concrete interval example retains the original amplitude prior.

Data availability. The exact rational verification programs and companion heat-trace manuscript are included in the accompanying source package.

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