Asymptotically optimal bounds for permanental cofactor matrices

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Abstract

For a complex correlation matrix B of order n, let $\mathcal{C}(B)_{ij} = \operatorname{per} B(i \mid j)$ be its raw permanental cofactor matrix. We bound the optimal constant γ_n in $\mathcal{C}(B) \preceq \gamma_n(\operatorname{per} B)I$. An elementary isolated-row induction and an explicit positive-definite equicorrelation family give, for every $n \geq 2$,

$$\frac{(n+1)^2}{4n} < \gamma_n \leq \frac{n^2+2n+3}{4n}.$$

Consequently $\gamma_n = n/4 + 1/2 + \Theta(1/n)$: both the leading coefficient and the constant term are sharp. The upper estimate extends, with the necessary diagonal weights, to all Hermitian positive semidefinite matrices, including zero diagonals. We also obtain an explicit nonnegative reserve for the standard natural-representation block in order six, and give an exact order-six obstruction to the raw cofactor bound with constant two. The proof is analytic and does not require finite polynomial certificates.

Keywords. Permanent; permanental cofactors; correlation matrices; positive semidefinite matrices; asymptotically optimal bounds.
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1 Introduction and normalization

If $A = (a_{ij})$ has order n, its permanent is

$$\operatorname{per} A = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Write $A(i \mid j)$ for the matrix obtained by deleting row i and column j, and set $\operatorname{per}(\varnothing) = 1$. Throughout, positive semidefinite means Hermitian positive semidefinite. A correlation matrix is such a matrix with every diagonal entry equal to one. Our cofactor convention is

$$\mathcal{C}(A)_{ij} = \operatorname{per} A(i \mid j). \tag{1}$$

There are no alternating signs. Some authors call the transpose of (1) the permanental adjoint; keeping this convention fixed is important for entry-weighted matrices and classical comparisons.

Positivity of permanental cofactor matrices is classical. It follows from an induced symmetric-power compression, or from products of Gram linear forms; see Marcus–Merris [4], Bhatia–Grover–Jain [2], and Tabata [7]. These constructions do not by themselves determine the sharp permanent-relative scalar bound under correlation normalization. The classical Marcus–Merris comparison uses the full original matrix as its metric. For positive definite A, their transposed-adjoint convention

gives $\mathcal{C}(A)^T \preceq n(\operatorname{per} A)A^{-1}$. Our estimate instead uses only the original diagonal, and remains uniform at singular correlation matrices.

The immediate methodological predecessor is Section 4 of Lavi [3], whose isolated-row cofactor argument yields a constant-two bound in order four and controls the natural Fourier block in order five. We use that mechanism in an all-order induction with its own initial condition. The resulting argument does not use the finite certificates appearing elsewhere in that work. An elementary lower family then shows that the induction has the correct linear and constant terms.

The recent theorem of Aouichaoui and Zhang [1] concerns the entry-weighted cofactor matrix $H(A) = A \circ \mathcal{C}(A)$. It bounds its quadratic form on each coordinate difference $e_i - e_j$ by $2 \operatorname{per} A$, thereby answering the question of Pate discussed there. Here the principal object is the raw matrix $\mathcal{C}(B)$: the bound holds in every direction under correlation normalization. The distinction matters even for positive equicorrelation matrices, where $H(B)$ has constant row sum $\operatorname{per} B$, while the raw cofactor ratio in Theorem 3.1 grows linearly with the order.

Our main results pair the recursive upper bound in Theorem 2.2 with the explicit lower family in Theorem 3.1, determining the linear and constant asymptotic terms. The weighted PSD extension and the two order-six consequences retain their respective diagonal and entry-weighted normalizations.

This is a different normalization and question from the two scalar cofactor comparisons shown to admit no uniform constant in [6], and from the joint occupation and compensation bounds for coupled cofactors in [5]. All proofs needed for the claims here are included below.

2 An all-order upper bound

Let $\mathcal{E}_n$ be the compact set of complex correlation matrices of order n. Define

$$\gamma_n = \max_{B \in \mathcal{E}_n} \frac{\lambda_{\max}(\mathcal{C}(B))}{\operatorname{per} B}. \tag{2}$$

The denominator is strictly positive, including at singular B, as the next lemma explains. Thus the maximum exists by continuity and compactness.

Lemma 2.1. *If $A \succeq 0$, then $\mathcal{C}(A) \succeq 0$. If A has positive diagonal, then* $\operatorname{per} A > 0$.

Proof. Realize A as the Gram matrix of linear forms $\ell_1, \dots, \ell_n$. Equip polynomials with the Fischer inner product, in which distinct monomials are orthogonal and $\|z^\alpha\|^2 = \alpha!$. The usual product expansion shows that the inner product of products of Gram forms is the permanent of their cross-Gram matrix. The deleted products $\prod_{k \neq i} \ell_k$ therefore have Gram matrix $\mathcal{C}(A)$ or $\mathcal{C}(A)^T$, depending on the inner-product convention. Transposing a Hermitian PSD matrix preserves positivity, proving the first assertion. If every diagonal entry is positive, each ℓ_i is nonzero. Their product is nonzero in the polynomial ring, so its squared Fischer norm, which equals $\operatorname{per} A$, is strictly positive. □

Theorem 2.2. *Set $c_1 = 1$ and, for $n \geq 2$, define*

$$c_n = \frac{1 + \sqrt{1 + (n-1)c_{n-1}}}{2}, \qquad u_n = \frac{n + 2 + 3/n}{4}. \tag{3}$$

For every complex correlation matrix B of order n,

$$\mathcal{C}(B) \preceq c_n(\operatorname{per} B)I \preceq u_n(\operatorname{per} B)I. \tag{4}$$

In particular $\gamma_n \leq c_n \leq u_n$.

Proof. The assertion with c_1 is immediate. Isolate an arbitrary last row of a correlation matrix of order $n \geq 2$:

$$B = \begin{pmatrix} T & z \\ z^* & 1 \end{pmatrix}, \qquad R = \mathcal{C}(T)^T, \quad p = \operatorname{per} T, \quad q = z^* R z.$$

Here $p > 0$ and $q \geq 0$ by Lemma 2.1. Direct permanent expansions give

$$\operatorname{per} B = p + q, \qquad \mathcal{C}(B)_{nn} = p, \qquad \mathcal{C}(B)_{nj} = (Rz)_j \quad (j < n). \tag{5}$$

For the last identity, expand the appended column of the minor deleting row n and column j. Its coefficient of z_i is $\operatorname{per} T(i \mid j) = R_{ji}$. Expansion in the last row then gives the first identity. This also verifies the complex orientation in (5).

Inductively $0 \preceq R \preceq c_{n-1} p I$, and hence $R^2 \preceq c_{n-1} p R$. Cauchy–Schwarz for the ordinary Euclidean norm now yields

$$\begin{aligned} p + \|Rz\|_1 &\leq p + \sqrt{n-1}\, \|Rz\|_2 \\ &\leq p + \sqrt{(n-1) c_{n-1} p q}. \end{aligned}$$

For $a \geq 0$, $p > 0$ and $q \geq 0$,

$$\frac{p + \sqrt{apq}}{p+q} \leq \frac{1 + \sqrt{1+a}}{2}. \tag{6}$$

Indeed the left side is the Rayleigh quotient of $\left(\begin{smallmatrix} 1 & \sqrt{a}/2 \\ \sqrt{a}/2 & 0 \end{smallmatrix}\right)$ at $(\sqrt{p}, \sqrt{q})$. With $a = (n-1)c_{n-1}$, every absolute row sum of $\mathcal{C}(B)$ is at most $c_n \operatorname{per} B$. Its largest eigenvalue is bounded by its largest absolute row sum, proving the induction. This step uses neither an inverse nor a lower bound for the smallest eigenvalue of B.

Finally $c_1 = 1 \leq u_1 = 3/2$. For $n \geq 2$,

$$1 + (n-1)u_{n-1} = \frac{n^2+6}{4}, \qquad \frac{1 + \sqrt{1 + (n-1)u_{n-1}}}{2} = \frac{1}{2} + \frac{\sqrt{n^2+6}}{4} \leq u_n,$$

because $\sqrt{n^2+6} \leq n + 3/n$. Monotonicity of the square-root map proves $c_n \leq u_n$ by induction. □

Corollary 2.3. *For every PSD matrix A of order n, put $\Delta = \operatorname{diag}(\sqrt{a_{11}}, \ldots, \sqrt{a_{nn}})$. Then*

$$\Delta \mathcal{C}(A) \Delta \preceq c_n (\operatorname{per} A) I \preceq u_n (\operatorname{per} A) I. \tag{7}$$

All zero-diagonal and rank-changing boundaries are included.

Proof. First suppose the diagonal is positive, write $A = \Delta B \Delta$, and put $d = \prod_i a_{ii}$. Diagonal scaling of each permanent gives

$$\operatorname{per} A = d \operatorname{per} B, \qquad \mathcal{C}(A) = d \Delta^{-1} \mathcal{C}(B) \Delta^{-1}.$$

Apply Theorem 2.2. The general assertion follows by applying the positive-diagonal result to $A + \varepsilon I$ and taking $\varepsilon \downarrow 0$. All entries and weights are continuous. □

The factors Δ in (7) are necessary. For instance, when $A = tI$, $\mathcal{C}(A) = t^{n-1} I$ and $\operatorname{per} A = t^n$; no fixed unweighted constant can work as $t \downarrow 0$.

3 A lower family and sharp two-term asymptotics

Let $J = \mathbf{1}\mathbf{1}^*$, and define $S_n(t) = \sum_{m=0}^{n} t^m/m!$.

Theorem 3.1. *For $n \geq 2$, set*

$$a = \frac{n-1}{n+1}, \qquad b = \frac{2}{n+1}, \qquad t = \frac{a}{b} = \frac{n-1}{2}, \qquad B_n = aI + bJ.$$

This positive-definite correlation matrix satisfies

$$\frac{\lambda_{\max}(\mathcal{C}(B_n))}{\operatorname{per} B_n} = \frac{(n+1)^2}{4n} + \frac{n^2-1}{4n}\frac{t^n/n!}{S_n(t)} > \frac{(n+1)^2}{4n}. \tag{8}$$

Consequently, for every $n \geq 2$,

$$\frac{n+2+1/n}{4} < \gamma_n \leq \frac{n+2+3/n}{4}. \tag{9}$$

Proof. Choosing the entries contributed by aI in the permanent expansion gives

$$\operatorname{per}(aI + bJ) = \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m}(n-m)! = n! b^n S_n(a/b). \tag{10}$$

All cofactors are nonnegative and every cofactor row has the same sum. The ones vector is therefore a top eigenvector. The sum of all cofactors is the derivative of the permanent in the J direction, so its row sum is $n^{-1}\partial_b \operatorname{per}(aI + bJ)$, with a held fixed. Since $S_n' = S_{n-1}$, division by (10) gives

$$\frac{\lambda_{\max}(\mathcal{C}(B_n))}{\operatorname{per} B_n} = \frac{1}{b}\left(1 - \frac{t}{n}\frac{S_{n-1}(t)}{S_n(t)}\right).$$

Substitute $S_{n-1} = S_n - t^n/n!$ and the chosen b, t to obtain (8). The extra term is positive because $t > 0$. Combine with Theorem 2.2. □

Corollary 3.2. *As $n \to \infty$,*

$$\gamma_n = \frac{n}{4} + \frac{1}{2} + \Theta(1/n).$$

The explicit upper constant satisfies

$$1 \leq \frac{u_n}{\gamma_n} < 1 + \frac{2}{(n+1)^2} \quad (n \geq 2).$$

Proof. The first statement is the interval (9). For the second, divide u_n by its strict lower endpoint $(n+1)^2/(4n)$. □

The chosen lower-family member need not maximize the ratio. At order two, for $B = \left(\begin{smallmatrix} 1 & z \\ \bar{z} & 1 \end{smallmatrix}\right)$, the cofactor ratio is $(1+|z|)/(1+|z|^2)$, whose maximum is $(1+\sqrt{2})/2$ at $|z| = \sqrt{2}-1$. Thus $\gamma_2 = c_2$. For general n, neither the exact γ_n nor the coefficient of $1/n$ in its asymptotics is determined by (9).

4 A quantified natural-block inequality in order six

Raw cofactors must be distinguished from their entry-weighted version

$$H(A) = A \circ \mathcal{C}(A), \qquad H(A)_{ij} = a_{ij} \operatorname{per} A(i \mid j). \tag{11}$$

The Schur product theorem makes $H(A)$ PSD, and row expansion gives $H(A)\mathbf{1} = (\operatorname{per} A)\mathbf{1}$. Thus $\mathbf{1}^{\perp}$ is invariant. We write $H_0(A)$ for the restriction to that subspace, with its inherited Euclidean inner product.

For completeness, if $\rho(\pi)e_j = e_{\pi(j)}$ is the natural permutation representation and $a_\pi = \prod_i a_{i,\pi(i)}$, then $\sum_\pi a_\pi \rho(\pi) = H(A)^T$. Its nontrivial natural block is the standard representation, labeled $(n-1,1)$. Transposition does not change the Frobenius norm. The statement below can therefore be read either as an elementary matrix inequality or as a Fourier-block bound.

Theorem 4.1. *For every PSD matrix A of order six, let $P = \operatorname{per} A$ and set*

$$p_i = a_{ii} \operatorname{per} A(i \mid i), \qquad q_i = P - p_i.$$

Then $p_i, q_i \geq 0$ and

$$5P^2 - \|H_0(A)\|_{\mathrm{F}}^2 \geq \sum_{i=1}^{6} \left(q_i^2 + \frac{p_i q_i}{10} \right) \geq 0. \tag{12}$$

Proof. First take a correlation matrix and isolate row i as in (5). The off-diagonal entries of that row of H are $\bar{z}_j (Rz)_j$. Every $|z_j| \leq 1$ by a principal 2-by-2 minor, so

$$\sum_j |H_{ij}|^2 \leq p_i^2 + z^* R^2 z \leq p_i^2 + c_5 p_i q_i.$$

Here $q_i = z^* R z \geq 0$. Theorem 2.2 gives $c_5 \leq u_5 = 19/10$. Subtracting the last bound from $P^2 = (p_i + q_i)^2$ yields

$$P^2 - \sum_j |H_{ij}|^2 \geq q_i^2 + p_i q_i / 10.$$

Sum over the six rows. Since H is Hermitian with eigenvalue P on the constant subspace, $\|H\|_{\mathrm{F}}^2 = P^2 + \|H_0\|_{\mathrm{F}}^2$, proving (12). For positive-diagonal $A = \Delta B \Delta$, each of H, P, p_i and q_i is multiplied by the same factor $\prod_j a_{jj}$ relative to its correlation-normalized version. The inequality transfers by homogeneity. Continuity from $A + \varepsilon I$ handles zero diagonals. □

This extends the isolated-row natural-block method of [3] to order six and retains an explicit reserve. It controls the $(5,1)$ block only.

5 An exact obstruction to a constant-two raw bound

Proposition 5.1. *The inequality $\mathcal{C}(B) \preceq 2(\operatorname{per} B) I$ fails for positive-definite correlation matrices of order six. It already fails at $B = (5/7)I + (2/7)J$.*

Proof. Theorem 3.1 gives a ratio greater than $49/24 > 2$. Direct evaluation, also obtainable from (10) and its derivative, supplies the exact values

$$\operatorname{per} B = \frac{553405}{117649}, \qquad \lambda_{\max}(\mathcal{C}(B)) = \frac{164665}{16807}, \qquad \frac{\lambda_{\max}(\mathcal{C}(B))}{\operatorname{per} B} = \frac{230531}{110681}.$$

Their difference is $\lambda_{\max}(\mathcal{C}(B)) - 2 \operatorname{per} B = 45845/117649 > 0$. □

This example has a useful centered realization. With $s = \sqrt{2}$, $W = 6I - J$ and $K = (5s/17)W$, the matrix K is PSD of rank five and $K\mathbf{1} = 0$. Moreover

$$K + sJ = \frac{30s}{17}I + \frac{12s}{17}J, \qquad \operatorname{diag}(K + sJ) = \frac{42s}{17}I.$$

Its correlation normalization is exactly B. Thus the raw constant-two comparison can fail on a rank-five centered residual base, not only on an unrelated correlation model. This obstruction concerns that raw comparison; it does not contradict Theorem 4.1 or disprove Chollet's inequality.

6 Further questions

The bounds determine the first two terms of the optimal correlation cofactor constant, but leave an interval of width $1/(2n)$. Determining the $1/n$ coefficient, the exact finite-order constants, or a rigidity description of near-extremizers would sharpen this extremal problem. The lower family is explicit and positive definite, whereas the upper proof allows arbitrary complex phases and all singular boundaries. Any sharper argument must retain enough of that generality to close the remaining interval. The natural-block reserve is a separate consequence; its usefulness in a coupled permanent inequality would require an additional, explicitly normalized coupling argument.

Methods and assistance. AI tools assisted with derivation, exposition and code preparation. The author is responsible for the mathematical claims, proofs and references. The theorems follow from the analytic arguments printed here, rather than from numerical optimization, finite sampling or unreproduced polynomial certificates. The accompanying source package contains a standard-library Python checker, `verify.py`, and exact output `verification.json`. It reconstructs complex rational Gram examples, row identities, diagonal congruences, the six-row reserve and the explicit lower family. These finite controls are reproducibility checks, not replacements for the analytic proofs above.

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