Arithmetic cutoff bounds and sharp tail capacity for local Nyman–Beurling certificates

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Abstract

We approximate the complete local Nyman–Beurling projection problem uniformly before optimizing its Dirichlet-polynomial coefficients. A support-sensitive tail estimate retains the shared arithmetic head, the height-dependent mixed Bernoulli covariances, both exact coefficient constraints and the entire norm and mean errors. Its error depends on the separation of the support's divisor frequencies. Dense supports give the regime $N = o(T)$; finite unions of fixed geometric lattices admit an $O(T^{-1})$ optimized-gain error even when $N = \exp(T^{\beta})$ for any fixed $\beta > 0$. Model maximizers then have an explicit bound on their loss in true gain. In contrast, the best true gain over each fixed geometric module, allowing arbitrary degrees and height-dependent coefficients, becomes smaller than every positive threshold on positive-lower-density sets of heights. Joint Hardy-kernel projection also bounds the complex defect required to escape such modules. Thus accurate finite models and adequate approximation dictionaries are separate requirements. The proof combines classical separated-frequency estimates with varying primitive amplitudes, an exact cotangent tail form and a compensated arithmetic mean.

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1 The local arithmetic problem

How accurately can a finite arithmetic calculation recover a local Nyman–Beurling projection gain when the height and permitted support grow together? The optimization uses an infinite Poisson norm. An error bound for one trial, or for its tail alone, need not control the optimized problem. We give a finite model whose complete norm and mean errors are uniform on the original constrained coefficient space. The resulting theorem controls both the optimal gain and the loss incurred by selecting a model maximizer.

The principal estimate depends on the arithmetic geometry of the permitted support, through the separation of its reduced divisor frequencies. It recovers $N = o(T)$ for dense supports and permits much larger geometric dictionaries. This extension exposes a complementary obstruction:

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even arbitrarily deep trials in each fixed finite geometric module have arbitrarily small best gain on positive-lower-density sets of heights. The model can therefore approximate the restricted optimization accurately while the dictionary itself remains inadequate for a positive all-height gain. A joint Hardy-kernel bound quantifies the complex evaluation defect needed to leave that restriction.

The local Hardy projection principle is classical [1], as are fractional-part gcd covariance [2], Jordan factorization [3], Bernoulli summation [4], high-parameter truncation [5, 6], absolute separated-frequency estimates [7, 8] and zeta universality [9]. The unshifted wavelet-tail sandwich of [10, Theorem 4] is the closest large-sieve predecessor; [11, 12] also give the relevant low-order covariance and primitive-shell identities. The additional estimates here account for the varying complex primitive amplitudes, the original Poisson weight and the compensated mean before optimization. Restricting the frequency set uses the classical separation parameter, rather than a new large-sieve inequality. Collision-uniform zero-packet kernel asymptotics [13] concern a different approximation problem.

Fix throughout

$$0 < a < \tfrac{1}{2}, \quad \alpha = \tfrac{1}{2} + a, \quad \sigma = \tfrac{1}{2} - a, \quad \lambda = \alpha + iT, \quad q = \sigma + iT, \quad T \in \mathbb{R}. \tag{1}$$

Let $\mathcal{A}_N$ be the Dirichlet polynomials

$$A(s) = \sum_{d \le N} c_d d^{-s}, \qquad A(1) = \sum_{d \le N} \frac{c_d}{d} = 0.$$

The coefficients may be complex and may depend on T. The integer N is a support cutoff, not the number of nonzero terms. Put

$$d_n = \sum_{d \mid n} c_d, \qquad B_n = \sum_{j \le n} d_j j^{-q}, \qquad p_n = n^{-2a} - (n+1)^{-2a}, \tag{2}$$

where $c_d = 0$ beyond the support. Define

$$m(A) = \sum_{n \ge 1} p_n B_n, \qquad Q(A) = \sum_{n \ge 1} p_n |B_n|^2, \qquad g(A) = \frac{|m(A)|^2}{Q(A)} \quad (A \ne 0),$$

and $g_N(\lambda) = \sup_{A \in \mathcal{A}_N \setminus \{0\}} g(A)$. For a trivial coefficient space the supremum is defined to be zero. For every integer $M \ge 1$, put $K = M + 1 \ge 2$ and define the finite heads

$$m_M = \sum_{n \le M} p_n B_n, \qquad Q_M = \sum_{n \le M} p_n |B_n|^2.$$

Proposition 1.1 (Poisson norm and mean identities)**.** *For every anchored finite polynomial, all these moments exist and*

$$m(A) = \zeta(\lambda) A(\lambda), \tag{3}$$

$$Q(A) = \frac{a}{\pi} \int_{\mathbb{R}} \frac{|\zeta(1/2 + it) A(1/2 + it)|^2}{a^2 + (t - T)^2} \, \mathrm{d}t. \tag{4}$$

For $A \ne 0$, $Q(A) > 0$ and $0 \le g(A) \le 1$.

Proof. The summatory divisor sequence satisfies

$$D(x) = \sum_{j \le x} d_j = \sum_d c_d \lfloor x/d \rfloor = -\sum_d c_d \{x/d\}, \qquad |D(x)| \le S_0 := \sum_d |c_d|.$$

Partial summation shows that $\sum d_n n^{-s}$ converges on $\Re s > 0$, agrees there by continuation with $\zeta(s)A(s)$, and that B_n is bounded at fixed q. Thus both moments converge. Summation by parts against p_n gives

$$\sum_{n\le M} p_n B_n = \sum_{j\le M} d_j j^{-(q+2a)} - (M+1)^{-2a} B_M.$$

The last term tends to zero, proving (3) since $q + 2a = \lambda$.

Initially on $\Re s > 1$, the Mellin identity is

$$\int_1^\infty B_{\lfloor x\rfloor} x^{q-s} \frac{\mathrm{d}x}{x} = \frac{\zeta(s)A(s)}{s-q}.$$

Since B_n is bounded, the integral is absolutely convergent for $\Re s > \sigma$; analytic continuation therefore identifies it there. Mellin Plancherel on the boundary gives

$$\frac{1}{2\pi}\int_{\mathbb{R}} \left|\frac{\zeta(1/2+it)A(1/2+it)}{1/2+it-q}\right|^2 \mathrm{d}t = \int_1^\infty |B_{\lfloor x\rfloor}|^2 x^{-2a}\frac{\mathrm{d}x}{x} = \frac{Q(A)}{2a},$$

which is (4). If $Q = 0$, positivity of p_n forces every B_n, hence every d_n and c_n, to vanish. Finally $\sum p_n = 1$, so Cauchy–Schwarz proves $|m|^2 \le Q$. □

Minimizing $\sum p_n|1 - zB_n|^2$ over $z \in \mathbb{C}$ gives $1 - g(A)$. This identifies g_N as an optimized projection gain, without an independence assumption on a chosen trial dictionary.

1.1 Periodic profiles and divisor coordinates

Write $B_j(x)$ for the Bernoulli polynomials, with $B_1(x) = x - 1/2$, and set

$$H_{r,d}(x) = -\frac{d^r}{(r+1)!} B_{r+1}(\{x/d\}), \quad r \ge 0, \qquad H_{r,c} = \sum_d c_d H_{r,d}. \tag{5}$$

In particular $H_{0,d}(n) = 1/2 - \{n/d\}$ uses the inclusive arithmetic value at an integer seam. For $r \ge 1$ the profiles are continuous and $H'_{r,d} = H_{r-1,d}$ almost everywhere. Put

$$t_k = \sum_{k|d} \frac{c_d}{d}, \qquad J_s(k) = \sum_{d|k} \mu(k/d) d^s.$$

Finite Möbius inversion gives $c_d = d\sum_{d|k} \mu(k/d)t_k$ and

$$A(s) = \sum_{k\le N} t_k J_{1-s}(k), \qquad t_1 = A(1) = 0. \tag{6}$$

The divisor coordinates in (6) will also diagonalize the complete mixed-order source. The exact classical variance and its full fixed-trial remainder are proved in Theorem E.1.

1.2 Principal consequence and proof structure

For a declared finite support $\mathcal{S} \subset \{1, \dots, N\}$ with $N = \max \mathcal{S} \ge 2$, put

$$\mathcal{D} = \{k \ge 2 : k \mid d \text{ for some } d \in \mathcal{S}\}, \qquad C_{\mathcal{S}} = 12 \sum_{k\in\mathcal{D}} \frac{\sigma_1(k)^2}{J_2(k)}, \quad \sigma_1(k) = \sum_{d|k} d.$$

Let $B_{\mathcal{S}}$ be the reciprocal minimum circular separation of the distinct reduced fractions v/k, $k \in \mathcal{D}$, $1 \le v < k$, $(v,k) = 1$; set it to one for at most one frequency. Then $B_{\mathcal{S}} \le \min(N^2, \operatorname{lcm}(\mathcal{S}))$. All optimizations below retain

$$\mathcal{H}_{\mathcal{S}} = \{c : \operatorname{supp} c \subset \mathcal{S},\ A_c(1) = A_c(q) = 0\}.$$

The second null supplies the quantitative tail expansion. The costed adaptation in Theorem C.1 relates it to the anchored trial space.

Theorem 1.2 (Uniform recovery of the constrained optimum)**.** *Fix $a \in (0, 1/2)$ and $\gamma > 1/(2\pi)$. As $T \to \infty$, permit finite supports $\mathcal{S} = \mathcal{S}(T)$ with $\log(2 + C_{\mathcal{S}}) = o(T)$ and $B_{\mathcal{S}} = o(NT)$. Assume $\mathcal{H}_{\mathcal{S}}$ is nontrivial. Set*

$$\theta = \frac{1 + 1/(2\pi\gamma)}{2}, \quad K = \lceil \gamma NT \rceil, \quad R = \max\left\{1, \left\lceil \frac{\log(2 + C_{\mathcal{S}}) + 3\log T}{\log(1/\theta)} \right\rceil \right\}.$$

Let $b = b_M + t_{R,K}$ be the corrected mean row (27), and use the positive cotangent weights W_k of (17) to form

$$\widehat{Q}(c) = Q_M(c) + \frac{2a}{K} \sum_{k \in \mathcal{D}} W_k(K/T)|t_k|^2.$$

For the true and model optima on this same space,

$$G_{\mathrm{true},\mathcal{S}} = \sup_{c \ne 0} \frac{|m(c)|^2}{Q(c)}, \qquad G_{b,\mathcal{S}} = \sup_{c \ne 0} \frac{|bc|^2}{\widehat{Q}(c)},$$

one has

$$|G_{\mathrm{true},\mathcal{S}} - G_{b,\mathcal{S}}| = O_{a,\gamma}\left(\frac{B_{\mathcal{S}}}{NT} + \frac{1}{T}\right).$$

The same order bounds the loss in true gain of a model maximizer. For a fixed finite union $\bigcup_r \{d_r b^j : j \ge 0\}$, the error is $O_{a,\gamma,b,\boldsymbol{d}}(T^{-1})$ whenever $\log(2 + \log N) = o(T)$. This includes $N = \exp(T^\beta)$ for every fixed $\beta > 0$.

Proof. The support-sensitive norm estimate is Theorem 6.1; its schedule is Corollary 6.3. Combine that estimate with the compensated mean in Theorem 8.1 and the selection bound in Corollary 8.2. Corollary 6.4 proves the geometric-support assertion. □

The head still contains $K \asymp NT$ terms. The theorem extends the uniform analytic range; its error estimate does not reduce the cost of forming that head. Theorem 10.2 supplies the complementary dictionary obstruction, and Proposition 10.4 sharpens its escape condition. The main proof proceeds through the primitive expansion, divisor covariance, support-sensitive norm return and compensated mean. The appendices collect evaluation duals, exact tail capacity, null adaptation and alternative certificates used in the examples.

2 Complete periodic expansions and uniform walls

Put $(q)_0 = 1$, $(q)_r = q(q+1)\cdots(q+r-1)$. The Bernoulli identities and Fourier bounds used here are classical [4].

Lemma 2.1 (Exact finite primitive formula)**.** *If $A(1) = A(q) = 0$, then for every integer $R \geq 1$ and $n \geq 1$,*

$$B_n = n^{-q} \sum_{r=0}^{R} (q)_r n^{-r} H_{r,c}(n) - (q)_{R+1} \int_n^\infty H_{R,c}(x) x^{-q-R-1} \, \mathrm{d}x. \tag{7}$$

For $r \geq 1$,

$$|H_{r,c}(x)| \leq \frac{4N^r S_0}{(2\pi)^{r+1}}, \qquad |H_{0,c}(n)| \leq S_0/2.$$

Proof. The limiting Dirichlet sum is $\zeta(q)A(q) = 0$. Exact partial summation yields

$$B_n = D(n)n^{-q} - q \int_n^\infty D(x) x^{-q-1} \, \mathrm{d}x = H_{0,c}(n) n^{-q} - q \int_n^\infty H_{0,c}(x) x^{-q-1} \, \mathrm{d}x.$$

The constant $D - H_{0,c} = -\sum c_d/2$ cancels. Repeated integration by parts using $H'_{r,c} = H_{r-1,c}$ gives a plus sign at every endpoint and the final negative remainder in (7). All primitive boundaries at infinity vanish since $\sigma + r > 0$. Bernoulli's complete Fourier series gives $\|B_{r+1}\|_\infty/(r+1)! \leq 2\zeta(r+1)/(2\pi)^{r+1} < 4/(2\pi)^{r+1}$, proving the bounds. The endpoint $H_{0,c}(n)$ is not replaced by a Fourier midpoint. □

Lemma 2.2 (A declared pole-free wall)**.** *Choose $0 < \theta < 1$ and $R \geq 1$, put $L = |q| + R + 1$, and suppose*

$$K \geq \frac{NL}{2\pi\theta}. \tag{8}$$

Define

$$b_\theta = \tfrac{1}{2} + \frac{2}{\pi} \frac{\theta}{1-\theta}, \qquad e_\theta = \frac{2}{\pi}(|q| + 2)\theta^R. \tag{9}$$

For every $n \geq K$, the finite profile sum in (7) has modulus at most $b_\theta S_0$, and its entire remainder has modulus at most $e_\theta S_0 n^{-\sigma}$. Consequently

$$Q - Q_M \leq (b_\theta + e_\theta)^2 S_0^2/K, \quad |m - m_M| \leq (b_\theta + e_\theta) S_0 K^{-\alpha}.$$

Proof. Since $|(q)_r| \leq L^r$ for $r \leq R+1$, the endpoint sum is bounded by $S_0[1/2 + (2/\pi)\sum_{r=1}^R \theta^r] \leq b_\theta S_0$. The complete remainder, after division by $n^{-\sigma}$, is at most

$$\frac{2}{\pi} S_0 \frac{L}{\sigma + R} \left(\frac{LN}{2\pi n}\right)^R \leq e_\theta S_0,$$

using $L/(\sigma + R) \leq |q| + 2$. Sum against the complete p_n weights and use $\sum_{n \geq K} p_n = K^{-2a}$, $2\sigma + 2a = 1$, $\sigma + 2a = \alpha$. □

For example $\theta = 1/2$ and $2^R \geq 8(|q| + 2)$ give the clean scalar receiver $|B_n| < (5/4)S_0 n^{-\sigma}$ after $n \geq N(|q| + R + 1)/\pi$. A shorter alternative paired-wave argument gives $1/2 + 2/\pi$ already at $n \geq N|q|/\pi$; it is stated with proof in Appendix H. Qualitative height-uniform truncation at a near-linear wall is classical [5, 6]. Here its role is the complete double-null arithmetic norm; the higher-order expansion also supplies the mixed covariance and its sharp height-scale law.

3 Mixed-order covariance and the complete tail

For fixed R, q, K define the positive quadratic form

$$U_R(c) = \frac{2a}{K} \int_0^1 \operatorname{av}_n \left| \sum_{r=0}^{R} (q)_r K^{-r} x^r H_{r,c}(n) \right|^2 \mathrm{d}x. \tag{10}$$

The average may be taken over any common integer period. The computation below avoids enumeration of $\operatorname{lcm}(1, \dots, N)$.

Theorem 3.1 (Complete mixed-order tail)**.** *Under the exact double null and wall* (8)*, put* $b = b_\theta$*,* $e = e_\theta$ *and*

$$\delta_R = \frac{[4aN^2 + 4a(a+1)]b^2}{K^2} + \frac{2be + e^2}{K}. \tag{11}$$

Then the whole infinite tail satisfies

$$|Q - Q_M - U_R(c)| \le \delta_R S_0^2. \tag{12}$$

In particular $G_M + U_R + N\delta_R I$ *is a complete norm upper on* $\{c : A(1) = A(q) = 0\}$*, where* $c^* G_M c = Q_M$*.*

Proof. For fixed orders and support indices, the centered periodic product $H_{r,d}(n)\overline{H_{s,e}(n)} - \operatorname{av} H_{r,d}\overline{H_{s,e}}$ has period $\operatorname{lcm}(d, e) \le N^2$. Its partial sums over any interval are bounded by twice that period times the product of the profile suprema. Abel summation against n^{-2-r-s} therefore bounds its tail by the same constant times K^{-2-r-s}. Multiplying by $2a$ and summing all orders, all support pairs and their absolute coefficient products pays at most $4aN^2b^2S_0^2/K^2$.

The exact weight obeys $|p_n - 2an^{-2a-1}| \le a(2a+1)n^{-2a-2}$. Since $\sum_{n \ge K} n^{-3-r-s} \le 2K^{-2-r-s}$, replacing this weight costs at most $2a(2a+1)b^2S_0^2/K^2$. Replacing the remaining power sum by $\int_K^\infty x^{-2-r-s}\,\mathrm{d}x = K^{-1-r-s}/(r+s+1)$ costs at most $2ab^2S_0^2/K^2$. The two costs combine to $4a(a+1)b^2S_0^2/K^2$. The resulting main term is precisely (10), including every complex mixed-order contribution.

Finally Lemma 2.2 bounds the difference between the complete square and the finite-profile square by $(2be + e^2)S_0^2 n^{-2\sigma}$. Its full p_n sum is at most $(2be + e^2)S_0^2/K$. This proves the two-sided enclosure. The matrix upper follows from $S_0^2 \le N\|c\|_2^2$. □

3.1 An exact divisor diagonalization

For $k \ge 2$, $1 \le v < k$, $(v, k) = 1$, set

$$u_r(k, v) = -\frac{k^r}{(r+1)!} \sum_{j=0}^{k-1} B_{r+1}(j/k) e^{-2\pi i v j/k}.$$

Theorem 3.2 (Full mixed-order divisor weights)**.** *On the anchored space,*

$$U_R(c) = \frac{2a}{K} \sum_{k=2}^{N} W_{R,k}(q, K)|t_k|^2, \tag{13}$$

where the strictly positive weights are

$$W_{R,k} = \int_0^1 \sum_{\substack{1 \le v < k \\ (v,k)=1}} \left| \sum_{r=0}^{R} (q)_r K^{-r} x^r u_r(k, v) \right|^2 \mathrm{d}x > 0. \tag{14}$$

For $R = 0$ *they reduce to* $J_2(k)/12$*.*

Proof. If $k \mid d$, Bernoulli multiplication gives the conditional mean

$$\mathbb{E}[H_{r,d}(n) \mid n \bmod k = j] = -\frac{k^{r+1}}{d(r+1)!} B_{r+1}(j/k).$$

The normalized Fourier coefficient at the reduced frequency v/k is thus $u_r(k,v)/d$; if $k \nmid d$ it is zero. This includes the exact $r = 0$ jump convention. Consequently the coefficient of $H_{r,c}$ is $u_r(k,v)t_k$. Orthogonality of distinct reduced rational frequencies on a common period proves (13). Every zero-frequency coefficient is proportional to t_1 and vanishes by anchoring. Since $u_0(k,v) = (1-e^{-2\pi iv/k})^{-1} \neq 0$, the polynomial in x cannot vanish identically, proving strict positivity. Finally the classical cosecant-square sum gives $\sum_{(v,k)=1} |u_0(k,v)|^2 = J_2(k)/12$. □

This is an all-order use of established Bernoulli/Farey covariance and divisor grouping [10, 12]. The quadratic Bernoulli wave and its divisor energy already appear in [11, Theorem 1]; primitive-shell orthogonality and the first two Jordan weights are also given in [12, Theorem 1]. The order-zero gcd form and the first correction are analyzed in Theorem E.1 and Corollary 7.2. The mixed q-dependent receiver retains their complete interaction.

For exact real arithmetic, define the Ramanujan sum $c_k(j) = \sum_{h|\gcd(k,j)} h\mu(k/h)$ and $b_r(j) = -k^r B_{r+1}(j/k)/(r+1)!$. Then

$$F_{rs}(k) = \sum_{j,l=0}^{k-1} b_r(j) b_s(l) c_k(l-j), \qquad W_{R,k} = \sum_{r,s=0}^{R} \frac{(q)_r \overline{(q)_s}}{K^{r+s}(r+s+1)} F_{rs}(k).$$

Thus the entire positive weight can be assembled from rational covariance data and enclosed complex rising factorials.

4 Height-proportional tail capacity and resonance

The finite covariance weights have a closed generating function. For reduced v/k,

$$\sum_{r\geq 0} u_r(k,v) z^r = \frac{1}{1-e^{z-2\pi iv/k}}, \qquad |z| < 2\pi \min(v/k, 1-v/k). \tag{15}$$

To prove this, sum the Bernoulli generating function $te^{xt}/(e^t-1)$ over $x = j/k$ with the character $e^{-2\pi ivj/k}$, and put $t = kz$. The constant term vanishes because the character is nontrivial; the finite geometric sum cancels $e^{kz} - 1$. This also recovers the discrete value $u_0 = (1-e^{-2\pi iv/k})^{-1}$.

Theorem 4.1 (Complete height-scale law)**.** *Fix N and a. Suppose $T \to +\infty$, $K/T \to \kappa > N/(2\pi)$. Choose a fixed θ with $N/(2\pi\kappa) < \theta < 1$ and integers $R = O(\log(T+2))$ satisfying $\theta^R(|q|+2)^2 \leq 1/8$. Uniformly on the height-dependent double-null spaces with $\|c\|_2 \leq 1$,*

$$\frac{K}{2a}(Q - Q_M) - \sum_{k=2}^{N} W_k(\kappa)|t_k|^2 \longrightarrow 0, \tag{16}$$

where

$$\begin{aligned} W_k(\kappa) &= \int_0^1 \sum_{\substack{1\leq v<k \\ (v,k)=1}} |1 - e^{i(x/\kappa - 2\pi v/k)}|^{-2} \,\mathrm{d}x \\ &= -\frac{\kappa}{2} \sum_{d|k} \mu(k/d) d \cot \frac{d}{2\kappa}. \end{aligned} \tag{17}$$

All arithmetic indices in the complete tail are included in this limit.

Proof. The wall (8) holds eventually. For fixed N, Theorem 3.1 gives $KN\delta_R \to 0$: its periodic part is $O(N^3/K)$, while $e_\theta = O(1/T)$. For each fixed order, $(q)_r/K^r \to (i/\kappa)^r$. The complete geometric bound from Lemma 2.2 is uniform, because $LN/(2\pi K)$ is eventually below the fixed $\theta < 1$. Dominated convergence in the finite-dimensional source and (15) prove (16), uniformly in its whole coefficient unit ball.

Now $|1 - e^{iy}|^{-2} = \frac{1}{4}\csc^2(y/2)$. Integrate each term to obtain a difference of cotangents. The starting cotangents sum to zero by opposite-residue pairing. Finally Möbius inversion of $\sum_{j=0}^{d-1} \cot(y - \pi j/d) = d\cot(dy)$ gives (17). The zero frequency $k = 1$ has already been removed by the anchor. □

The theorem fixes N while height varies. For arbitrary moving support the quantitative recipient is the finite enclosure (12), with its displayed masses, wall and error; it is available before passing to any limit.

Corollary 4.2 (Higher-Jordan inflation)**.** *For $k > 1$ and $\kappa > k/(2\pi)$,*

$$W_k(\kappa) = \sum_{\ell \geq 1} \frac{\zeta(2\ell) J_{2\ell}(k)}{2^{2\ell-1}\pi^{2\ell}\kappa^{2\ell-2}} = \frac{J_2(k)}{12} + \frac{J_4(k)}{720\kappa^2} + \frac{J_6(k)}{30240\kappa^4} + \cdots . \tag{18}$$

In particular $W_k(\kappa) > J_2(k)/12$. If also $\kappa > k/4$,

$$W_k(\kappa) \leq \left(1 - \frac{k}{4\kappa}\right)^{-2} \frac{J_2(k)}{12}.$$

At the first resonance,

$$W_k(\kappa) \sim \frac{k^3}{8\pi^3(\kappa - k/(2\pi))} \quad \text{as } \kappa \downarrow k/(2\pi). \tag{19}$$

Proof. Use the convergent expansion $\cot y = y^{-1} - 2\sum_{\ell \geq 1} \zeta(2\ell) y^{2\ell-1}/\pi^{2\ell}$ in (17). The inverse-power term cancels because $\sum_{d|k} \mu(k/d) = 0$; the remaining divisor sums are exactly $J_{2\ell}(k) > 0$. For the upper bound, the triangle inequality gives

$$|1 - e^{i(x/\kappa - 2\pi v/k)}| \geq |1 - e^{-2\pi i v/k}| - 1/\kappa,$$

while $|1 - e^{-2\pi i v/k}| \geq 2\sin(\pi/k) \geq 4/k$. Compare termwise with the order-zero sum. At the boundary in (19), only the $d = k$ cotangent has a pole. Expanding its argument at $\kappa = k/(2\pi)$ gives the stated coefficient. □

The old fixed-parameter gcd coefficient is thus strictly smaller than the height-scale leading coefficient at every finite pole-free κ. Its original fixed-parameter theorem retains its remainder; the mixed receiver pays this additional contribution when height varies. Cotangent/Nyman–Beurling connections and Vasyunin sums have classical antecedents [14]; the present law is for the complete native tail with simultaneous height and wall variation.

4.1 Capacity in the resummed metric

For the finite weights $W_{R,k} > 0$, put $j_z(k) = J_{1-z}(k)$ and

$$\mathcal{D}_R = \sum_{k=2}^{N} \frac{|j_\lambda(k)|^2}{W_{R,k}} - \frac{\left|\sum_{k=2}^{N} j_\lambda(k)\overline{j_q(k)}/W_{R,k}\right|^2}{\sum_{k=2}^{N} |j_q(k)|^2/W_{R,k}}.$$

When the constraints are feasible and $\mathcal{D}_R > 0$, exact weighted projection gives

$$\inf_{\substack{A(1)=A(q)=0\\ A(\lambda)=1}} U_R(c) = \frac{2a}{K\mathcal{D}_R}.$$

Indeed (6) expresses the two rows in the diagonal W metric; project the λ row off the q row and apply Cauchy–Schwarz with equality at its representer. The q row is nonzero for $N \geq 2$, since $J_{1-q}(2) = 2^{1-q} - 1 \neq 0$.

Define $U_\infty(c) = (2a/K)\sum_{k=2}^{N} W_k(\kappa)|t_k|^2$. For $\kappa \geq N$, Corollary 4.2 gives

$$\frac{2a}{K}V(c) \leq U_\infty(c) \leq \frac{16}{9}\frac{2a}{K}V(c).$$

The corresponding constrained resummed capacities lie between the old variance capacity and $16/9$ times it. Near resonance, the explicitly growing weights instead identify the expensive divisor coordinates.

5 The variance-priced growing-support comparison

Throughout this section $N \geq 2$ is an integer, and coefficients have the native support $1, \ldots, N$. The preceding height-scale theorem fixes the support. We now give a quantitative comparison on varying support and height, paying the inverse divisor-coordinate cost explicitly. Write $\sigma_1(k) = \sum_{d|k} d$ and

$$w_0(k) = \frac{J_2(k)}{12}, \qquad V(c) = \sum_{k=2}^{N} w_0(k)|t_k|^2, \qquad C_N = 12\sum_{k=2}^{N}\frac{\sigma_1(k)^2}{J_2(k)}.$$

These quantities use the same t_k and original coefficients as the full native recipient.

Lemma 5.1 (Complete coefficient price)**.** *For every anchored vector,*

$$S_0^2 \leq C_N V(c), \qquad C_N \leq \frac{897}{896}\frac{\pi^6}{18}N. \tag{20}$$

Proof. Möbius inversion gives $c_d = d\sum_{d|k}\mu(k/d)t_k$. Since $t_1 = 0$, this implies $S_0 \leq \sum_{k=2}^{N}\sigma_1(k)|t_k|$; weighted Cauchy–Schwarz proves the first inequality, including the coefficient c_1.

For the second, put $f(k) = \sigma_1(k)^2/J_2(k)$, $f(1) = 1$, and $g = f * \mu$. Both are multiplicative. If $x = 1/p$ and $j \geq 1$, then

$$f(p^j) = \frac{(1-x^{j+1})^2}{(1-x)^2(1-x^2)}.$$

These values increase, including $f(p) > f(1)$, so $g \geq 0$. The local sum is

$$\begin{aligned} B_p = \sum_{j\geq 0} g(p^j)p^{-j} &= (1-x)\sum_{j\geq 0} f(p^j)x^j \\ &= \frac{1+x^2}{(1-x)^3(1+x)^2(1+x+x^2)} - \frac{x^2}{1+x}. \end{aligned}$$

Direct algebra gives

$$B_p(1-x^2)^3 = 1 + \frac{x^3}{1+x+x^2} - x^2(1-x)(1-x^2)^2.$$

For $p \geq 3$, $x \leq 1/3$ and

$$\frac{x}{1+x+x^2} \leq \frac{1}{3} < \frac{128}{243} \leq (1-x)(1-x^2)^2,$$

so $B_p \leq (1-p^{-2})^{-3}$. At $p=2$, $B_2 = 299/126 = (897/896)(1-1/4)^{-3}$. Nonnegative partial Euler products are therefore bounded by $(897/896)\zeta(2)^3$. Consequently

$$\sum_{k\leq N} f(k) = \sum_{d\leq N} g(d)\lfloor N/d \rfloor \leq N \sum_{d\geq 1} \frac{g(d)}{d} \leq \frac{897}{896}\zeta(2)^3 N.$$

Multiplying by 12 proves (20). Positivity also justifies the series and product limits used here. □

Lemma 5.2 (Comparison throughout the pole-free range)**.** *For every $k > 1$ and $\kappa > k/(2\pi)$,*

$$w_0(k) \leq W_k(\kappa) \leq \left(1 - \frac{k}{2\pi\kappa}\right)^{-2} w_0(k).$$

Proof. The lower bound is the positive Jordan series. For the upper, put $y = \pi v/k$ and $h = x/(2\kappa)$. Then $0 \leq h < \pi/k \leq y < \pi$. Concavity of sine on $[0,\pi]$ gives

$$\sin(y-h) \geq (1-h/y)\sin y \geq \left(1 - \frac{k}{2\pi\kappa}\right)\sin y.$$

Compare the squared reciprocal denominators in (17), integrate in x, and sum over the reduced residues. □

Theorem 5.3 (Finite relative cotangent comparison)**.** *Let $T \geq 1$, $R \geq 1$ and $0 < \theta < 1$, and require the declared wall* (8)*. Use δ_R from* (11) *with this same θ. Put*

$$L = |q| + R + 1, \qquad \kappa = K/T, \qquad Z = \left(1 - \frac{NT}{2\pi K}\right)^{-1},$$

$$\eta_R = 2\sqrt{2}\left(\frac{\theta}{L(1-\theta)^3} + \frac{\theta^{R+1}}{1-\theta}\right), \qquad \mathcal{H} = \eta_R(2Z + \eta_R) + \frac{K\delta_R C_N}{2a}.$$

Then, uniformly over the actual exact double-null coefficient space,

$$\left|\frac{K}{2a}(Q - Q_M) - \sum_{k=2}^{N} W_k(K/T)|t_k|^2\right| \leq \mathcal{H}V(c). \tag{21}$$

Every infinite arithmetic tail and mixed primitive order in this comparison is paid.

Proof. The wall implies $NT/(2\pi K) < 1$, so every cotangent weight and Z are finite. For $r \geq 1$, discrete Parseval applied to the unnormalized Fourier transform defining u_r gives

$$\left(\sum_{(v,k)=1} |u_r(k,v)|^2\right)^{1/2} \leq \frac{4k^{r+1}}{(2\pi)^{r+1}} \leq 2\sqrt{2}\left(\frac{k}{2\pi}\right)^r \sqrt{w_0(k)}.$$

Indeed the full Fourier sum is k times the sum of squares of the k samples, each bounded by $4k^r/(2\pi)^{r+1}$; restriction to reduced residues decreases it. The second inequality uses $J_2(k) \geq 6k^2/\pi^2$.

For $1 \le r \le R$, telescope the products $(q)_r$ and $(iT)^r$. Every individual factor has modulus at most L, while its difference from iT is $\sigma + j$. Consequently

$$|(q)_r - (iT)^r| \le L^{r-1} \sum_{j=0}^{r-1} (\sigma + j) \le \frac{r(r+1)}{2} L^{r-1}.$$

In the Hilbert space $L^2([0,1])$ with reduced-residue coordinates, compare the finite profile $\sum_{r=0}^{R} (q)_r K^{-r} x^r u_r(k,v)$ with its exact resummation $(1 - e^{iTx/K - 2\pi iv/k})^{-1}$. The $r = 0$ terms agree, including the original jump. Using $kL/(2\pi K) \le \theta$ and the complete identity

$$\sum_{r \ge 1} \frac{r(r+1)}{2} \theta^r = \frac{\theta}{(1-\theta)^3},$$

the finite coefficient discrepancy has norm at most $2\sqrt{2}\,\theta \sqrt{w_0(k)}/[L(1-\theta)^3]$. All omitted $r > R$ terms together cost at most $2\sqrt{2}\,\theta^{R+1} \sqrt{w_0(k)}/(1-\theta)$. The generating function (15) is used strictly inside its pole-free disk, so the total norm distance is at most $\eta_R \sqrt{w_0(k)}$.

Lemma 5.2 bounds the resummed norm by $Z\sqrt{w_0(k)}$. The difference of the squared norms is therefore at most $\eta_R(2Z + \eta_R) w_0(k)$. Sum against $|t_k|^2$ and add the complete scaled error $K\delta_R S_0^2/(2a)$ from Theorem 3.1. Lemma 5.1 proves (21). □

5.1 The variance-priced matrix option

Let C be the inverse divisor map $c = Ct$, restricted to $t_1 = 0$. Another complete positive upper for the original norm is

$$C^* G_M C + \frac{2a}{K} \operatorname{diag}(W_{R,k}) + \delta_R C_N \operatorname{diag}(w_0(k)).$$

The same exterior-null row remains imposed. This follows directly from $S_0^2 \le C_N V(c)$ and is an alternative whole quadratic envelope to the earlier coefficient-metric price. The corrected mean is still evaluated on the actual $c = Ct$; its error is not transferred to an unpaid identity metric.

Since $W_k \ge w_0(k)$, an error $\mathcal{H} < 1$ also gives a relative $(1 \pm \mathcal{H})$ enclosure around the cotangent tail. Adding the positive unchanged head preserves the same multiplicative enclosure for the full norm. On any nonempty family with additional exact linear constraints, taking infima preserves these bounds; no conditioning assumption on its evaluation rows is needed for that comparison. This supplies uniform moving-support norm control; a favorable corrected numerator and full-head gain still require their own certificates.

6 Support-sensitive tail estimates

The variance-priced comparison above bounds each periodic product separately. Applying the classical absolute Farey block inequality before squaring the varying primitive profile improves the moving-support regime. The absolute Hilbert and large-sieve input is Montgomery–Vaughan [7, Section 2]; the same sine-quotient argument is explained in [8, Section 17.3]. An unshifted Nyman–Beurling wavelet tail sandwich using primitive Farey frequencies, the absolute block estimate and Abel summation appears in [10, Theorem 4]. The return here retains the height-dependent rising factors, their varying amplitudes, the exact Poisson weight and every mixed primitive order of the original double-null recipient.

Theorem 6.1 (Support-sensitive complete tail estimate)**.** *Let $0 < a < 1/2$, $T \geq 1$, $N \geq 2$, $R \geq 1$ and $K = M + 1 \geq 2$. Use a declared support $\mathcal{S}$ and the parameters $B_{\mathcal{S}}, C_{\mathcal{S}}$ defined in Section 1. Require both exact nulls $A(1) = A(q) = 0$, fix $0 < \theta < 1$, and impose the declared wall $K \geq NL/(2\pi\theta)$, where $L = |q| + R + 1$. Use the same b_θ, e_θ as in (9), and put*

$$\begin{aligned}
B_0 &= 1 + \frac{2\sqrt{2}\theta}{1-\theta}, \qquad B_1 = \frac{2\sqrt{2}\theta}{(1-\theta)^2}, \qquad A_\theta = B_0^2 + B_0 B_1, \\
\Delta_{\theta,\mathcal{S}} &= \frac{2a(B_{\mathcal{S}}+1)A_\theta + a(2a+1)B_0^2}{K^2} + \frac{(2b_\theta e_\theta + e_\theta^2)C_{\mathcal{S}}}{K}, \\
\eta_\theta &= 2\sqrt{2}\left(\frac{\theta}{L(1-\theta)^3} + \frac{\theta^{R+1}}{1-\theta}\right), \\
Z &= \left(1 - \frac{NT}{2\pi K}\right)^{-1}, \\
H_{\theta,\mathcal{S}} &= \eta_\theta(2Z + \eta_\theta) + \frac{K\Delta_{\theta,\mathcal{S}}}{2a}.
\end{aligned}$$

For the full mixed form U_R of (10) and the actual cotangent form $W(c) = \sum_{k=2}^{N} W_k(K/T)|t_k|^2$, one has

$$|Q - Q_M - U_R(c)| \leq \Delta_{\theta,\mathcal{S}} V(c), \tag{22}$$

$$\left|\frac{K}{2a}(Q - Q_M) - W(c)\right| \leq H_{\theta,\mathcal{S}} V(c). \tag{23}$$

These finite complete-tail statements do not require $H_{\theta,\mathcal{S}} < 1$.

Proof. Index the distinct reduced frequencies with denominators $k \in \mathcal{D}$ by $\omega = v/k$. Write A_r for the vector with entries $u_r(k, v)t_k$. The exact discrete Fourier representation of the finite primitive profile is

$$f(n) = \sum_{\omega} b_\omega(n) e^{2\pi i \omega n}, \qquad b(x) = \sum_{r=0}^{R} (q)_r x^{-r} A_r.$$

Every order has zero constant Fourier component because $t_1 = 0$. The right-jump value at order zero is retained in u_0. Discrete Parseval, as in the proof of Theorem 5.3, gives

$$\|A_0\| = \sqrt{V(c)}, \qquad \|A_r\| \leq 2\sqrt{2}\left(\frac{N}{2\pi}\right)^r \sqrt{V(c)} \quad (r \geq 1).$$

Since $|(q)_r| \leq L^r$, the declared wall gives, for every $x \geq K$,

$$\|b(x)\| \leq B_0\sqrt{V}, \qquad \|b'(x)\| \leq B_1\sqrt{V}/x. \tag{24}$$

These frequencies have circular separation $B_{\mathcal{S}}^{-1}$ when there are at least two of them. With one frequency the off-diagonal block below is zero. Let E_n have entries $e^{2\pi i(\omega_j - \omega_i)n}$ and put $D_{K,J} = \sum_{n=K}^{J}(E_n - I)$. The absolute cosecant Hilbert inequality gives

$$\|D_{K,J}\|_{\mathrm{op}} \leq B_{\mathcal{S}} \quad (J \geq K). \tag{25}$$

For completeness, each off-diagonal geometric sum is a diagonal unitary conjugate of a sine quotient with denominator $\sin \pi(\omega_j - \omega_i)$. Splitting its sine numerator into two exponentials expresses the matrix as the difference of two unitary conjugates of the cosecant kernel, divided by $2i$. Apply the

absolute Hilbert bound to both terms. This proves both signs of the Hermitian operator estimate; a one-sided upper large sieve alone is not used as a two-sided estimate.

Apply matrix Abel summation to the exact original weight

$$w(x) = x^{-2\sigma}\big[x^{-2a} - (x+1)^{-2a}\big] = 2ax^{-2\sigma}\int_0^1 (x+t)^{-2a-1}\,dt.$$

It satisfies $0 < w(x) \le 2ax^{-2}$ and $|w'(x)| \le 4ax^{-3}$; the derivative uses $2\sigma + 2a = 1$. For $F(x) = w(x)b(x)b(x)^*$, (24) gives, whenever $\|D\| \le B_{\mathcal{S}}$,

$$|\operatorname{Tr}(DF'(x))| \le B_{\mathcal{S}}\big(|w'|\|b\|^2 + 2w\|b'\|\|b\|\big) \le 4aB_{\mathcal{S}}A_\theta V x^{-3}.$$

The Abel sum starts with $D_{K,K-1} = 0$; its boundary at infinity vanishes by these bounds. Integration over the entire half-line gives

$$\left|\sum_{n\ge K} w(n)(|f(n)|^2 - \|b(n)\|^2)\right| \le \frac{2aB_{\mathcal{S}}A_\theta V}{K^2}.$$

This pays variation of every amplitude, rather than freezing it at K.

Replacing the diagonal weight by $2an^{-2}$ costs at most $a(2a+1)B_0^2V/K^2$: use $|w(n) - 2an^{-2}| \le a(2a+1)n^{-3}$ and $\sum_{n\ge K} n^{-3} \le K^{-2}$ for $K \ge 2$. The complete sum/integral replacement for $h(x) = 2ax^{-2}\|b(x)\|^2$ costs at most $2aA_\theta V/K^2$, because $|h'(x)| \le 4aA_\theta V x^{-3}$. Its integral is exactly $U_R(c)$.

Finally the full primitive remainder and its cross term are bounded by $(2b_\theta e_\theta + e_\theta^2)S_0^2/K$. Here Lemma 2.2 uses the same general θ, and $\sum_{n\ge K} p_n n^{-2\sigma} \le K^{-1}$. Finite Möbius inversion gives $S_0 \le \sum_{k\in\mathcal{D}} \sigma_1(k)|t_k|$, including c_1; weighted Cauchy–Schwarz gives $S_0^2 \le C_{\mathcal{S}}V$. Apply this bound and add all payments to prove (22). The constants from a half-ratio wall cannot be retained unchanged when the declared ratio is enlarged.

The finite-profile/cotangent comparison in Theorem 5.3 gives the squared-norm error $\eta_\theta(2Z + \eta_\theta)V$. It retains all omitted orders and uses Lemma 5.2 throughout the pole-free range. Add the scaled error $K\Delta_{\theta,\mathcal{S}}V/(2a)$ to obtain (23). □

Corollary 6.2 (Dense growing support)**.** *Take $\mathcal{S} = \{1,\dots,N\}$, so that $B_{\mathcal{S}} \le N^2$ and $C_{\mathcal{S}} = C_N$. Fix $a \in (0,1/2)$ and $\gamma > 1/(2\pi)$. Set*

$$\theta = \frac{1 + 1/(2\pi\gamma)}{2}, \qquad K = \lceil \gamma NT \rceil, \qquad R = \max\left(1, \left\lceil \frac{3\log T}{\log(1/\theta)} \right\rceil\right).$$

Eventually the wall is admitted uniformly in N and

$$H_{\theta,\mathcal{S}} = O_{a,\gamma}(N/T + 1/T).$$

Thus $N = o(T)$ gives a uniform complete cotangent tail law on the actual varying double-null spaces after normalization $V = 1$. The constants need not remain bounded as the fixed gap $\gamma - 1/(2\pi)$ shrinks.

Proof. Here $L/T \to 1$, $\theta^R \le T^{-3}$, $e_\theta = O(T^{-2})$, $\eta_\theta = O(T^{-1})$ and $Z = O_\gamma(1)$. Since $C_N = O(N)$ and $K \asymp_\gamma NT$, $K\Delta_{\theta,\mathcal{S}} = O_{a,\gamma}(N/T + N/T^2)$. This proves the asserted bound and its sufficient regime. □

Since $W \geq V$, whenever $H_{\theta,\mathcal{S}} < 1$ the model $\widehat{Q} = Q_M + (2a/K)W$ satisfies

$$(1 - H_{\theta,\mathcal{S}})\widehat{Q} \leq Q \leq (1 + H_{\theta,\mathcal{S}})\widehat{Q}.$$

The same relative bounds hold after taking infima over any nonempty exact-null affine coefficient family, even if its infimum is zero. A complete finite mixed upper in inverse divisor coordinates $c = Ct$ is

$$C^* G_M C + \frac{2a}{K} \operatorname{diag}(W_{R,k}) + \Delta_{\theta,\mathcal{S}} \operatorname{diag}(J_2(k)/12),$$

with $t_1 = 0$ and the original exterior-null row imposed. The complete corrected mean and its error use the same C and actual Ct. This is a full receiving matrix, not merely a tail capacity. It does not itself assert favorable numerator size or positive all-height gain.

Corollary 6.3 (Uniform support-dependent gain approximation)**.** *Fix $a \in (0, 1/2)$ and $\gamma > 1/(2\pi)$, and assume $\mathcal{H}_{\mathcal{S}}$ is nontrivial. Set*

$$\theta = \frac{1 + 1/(2\pi\gamma)}{2}, \qquad K = \lceil \gamma N T \rceil, \qquad R = \max\left\{1, \left\lceil \frac{\log(2 + C_{\mathcal{S}}) + 3\log T}{\log(1/\theta)} \right\rceil \right\}.$$

If $\log(2 + C_{\mathcal{S}}) = o(T)$, the declared wall is eventually admitted and

$$H_{\theta,\mathcal{S}} = O_{a,\gamma}\left(\frac{B_{\mathcal{S}}}{NT} + \frac{1}{T}\right).$$

On the same exact double-null support subspace, the corrected mean has radius

$$e_{\mathcal{S}} = \frac{2\sqrt{2a}}{\pi}\sqrt{C_{\mathcal{S}}} K^{-a} \theta^R \left(2 + \frac{L}{R^2}\right) = O_{a,\gamma}(N^{-a} T^{-a-2}).$$

Whenever $H_{\theta,\mathcal{S}} < 1$, the optimized-gain bounds of Theorem 8.1 hold with $H = H_{\theta,\mathcal{S}}$ and $e = e_{\mathcal{S}}$. In particular, $B_{\mathcal{S}} = O(N)$ gives $|G_{\mathrm{true},\mathcal{S}} - G_{b,\mathcal{S}}| = O(1/T)$.

Proof. Here $R = o(T)$ and $L/T \to 1$. Also $\theta^R \leq (2 + C_{\mathcal{S}})^{-1} T^{-3}$, so $e_\theta = O((2 + C_{\mathcal{S}})^{-1} T^{-2})$, $\eta_\theta = O(T^{-1})$ and $Z = O_\gamma(1)$. The full remainder payment is $O(T^{-2})$, while the other scaled norm payments are $O(B_{\mathcal{S}}/(NT) + 1/(NT))$. This proves the norm estimate. In the compensated mean proof use $S_R \leq N^R \sqrt{C_{\mathcal{S}} V}$ and $\widehat{Q} \geq 2aV/K$. This gives the displayed radius and the same dual-norm argument on the same coefficient subspace. Its asymptotic bound follows from $L = O(T)$, $R \geq 1$ and $\sqrt{C_{\mathcal{S}}}/(2 + C_{\mathcal{S}}) \leq 1$. □

Corollary 6.4 (Large fixed-module dictionaries)**.** *Fix $b \geq 2$ and positive integers $d_1, \ldots, d_m$. Let $\mathcal{S}(T)$ be a finite subset of $\bigcup_r \{d_r b^k : k \geq 0\}$, with nontrivial double-null coefficient space, and write $N = \max \mathcal{S}(T) \geq 2$. If $\log(2 + \log N) = o(T)$, the schedule in Corollary 6.3 gives relative full-norm and optimized-gain errors $O_{a,\gamma,b,\boldsymbol{d}}(1/T)$.*

Proof. Put $D_0 = \operatorname{lcm}(d_1, \ldots, d_m)$ and let k_* be the largest exponent occurring in $\mathcal{S}$. Every support index divides $D_0 b^{k_*}$ and $b^{k_*} \leq N$, whence $B_{\mathcal{S}} \leq D_0 N$. Every prime dividing a member of $\mathcal{D}$ divides bD_0. The multiplicative formula for the coefficient price therefore gives

$$\frac{\sigma_1(k)^2}{J_2(k)} \leq \prod_{p \mid bD_0} \frac{1}{(1 - p^{-1})^2 (1 - p^{-2})} =: F_0 \quad (k \in \mathcal{D}).$$

Consequently

$$C_{\mathcal{S}} \leq 12 F_0 \tau(D_0 b^{k_*}) = O_{b,D_0}((1 + \log N)^{\omega(b)}).$$

The stated growth condition implies $\log(2 + C_{\mathcal{S}}) = o(T)$, and the preceding corollary applies. □

For example, every $N = \exp(T^\beta)$ with fixed $\beta > 0$ is admissible. These are uniform analytic estimates for finite models: the retained head still has $K \asymp NT$ terms. Together with Theorem 10.2, they separate reliable model approximation from dictionary adequacy. On a positive-lower-density set of heights the best gain of the entire fixed module is below any prescribed positive threshold, regardless of its geometric depth. On each admitted finite support, the corrected and true double-null optima differ by $o(1)$. The latter is at most the all-degree anchored module optimum, so these corrected models also have lower limit zero.

7 Compensated mean tails and actual alignment

The elementary mean bound (61) pays magnitudes separately. The two nulls permit a matched kernel that keeps their cancellation. Write $B_j(u)$ for the Bernoulli polynomials and $(z)_j$ for rising factorials. For integers $r \geq 1$ define

$$S_r = \sum_d d^r |c_d|, \quad C_r = \frac{\max_{0\leq u\leq 1} |B_{r+1}(u)|}{(r+1)!}, \quad \Delta_r = (\lambda)_r - (q)_r.$$

In particular $C_1 = 1/12$, $C_2 = \sqrt{3}/216$ and $C_3 = 1/720$. Bernoulli differentiation and multiplication used below are classical [4, Section 24.4].

Theorem 7.1 (Matched complete mean correction)**.** *Suppose $A(1) = A(q) = 0$. Put $K = M + 1$, where $M \geq 1$ is an integer. For each finite integer $R \geq 1$,*

$$m - m_M = t_{R,K}c + \mathcal{E}_{R,K}(c), \tag{26}$$

$$t_{R,K}c = \sum_{r=1}^{R} \frac{\Delta_r K^{-\lambda-r}}{(r+1)!} \sum_d c_d d^r B_{r+1}(\{K/d\}), \tag{27}$$

$$|\mathcal{E}_{R,K}(c)| \leq C_R S_R K^{-\alpha-R} \mathcal{A}_R, \tag{28}$$

$$\mathcal{A}_R = \frac{|\Delta_{R+1}|}{\alpha+R} + \frac{2a|(q)_{R+1}|}{(\sigma+R)(\alpha+R)}. \tag{29}$$

Every statement is for the complete infinite tail.

Proof. Set $F_K(x) = x^{-\lambda} - K^{-2a}x^{-q}$ for $x \geq K$. The summation identity in Proposition 1.1 and the zero limit of B_M under $A(q) = 0$ give

$$m - m_M = \sum_{j\geq K} d_j F_K(j) = -\int_K^\infty D(x)F_K'(x)\,\mathrm{d}x.$$

The Dirichlet tails converge since D is bounded and $\Re q > 0$. Crucially $F_K(K) = 0$, so there is no endpoint jump contribution, including when K is divisible by a support index. Also $F_K(\infty) = 0$. The continuous period mean of D is $\overline{D} = -\frac{1}{2}\sum_d c_d$, so its constant part contributes zero. For the centered function $h = D - \overline{D}$, put

$$P_r(x) = -\sum_d \frac{c_d d^r}{(r+1)!} B_{r+1}(\{x/d\}).$$

These are continuous periodic functions with $P_1' = h$ almost everywhere, $P_r' = P_{r-1}$ for $r \geq 2$, and $|P_r| \leq C_r S_r$. Repeated ordinary integration by parts yields

$$m - m_M = \sum_{r=1}^{R} (-1)^{r+1} P_r(K) F_K^{(r)}(K) + (-1)^{R+1} \int_K^\infty P_R(x) F_K^{(R+1)}(x)\,\mathrm{d}x.$$

Here $F_K^{(r)}(K) = (-1)^r \Delta_r K^{-\lambda-r}$, which proves the correction and its signs. To bound the remainder, scale $x = Ky$ and use the exact split

$$(\lambda)_{R+1} y^{-2a} - (q)_{R+1} = \Delta_{R+1} y^{-2a} + (q)_{R+1}(y^{-2a} - 1).$$

Against $y^{-\sigma-R-1}\,\mathrm{d}y$, its absolute upper bound integrates to

$$\frac{|\Delta_{R+1}|}{\alpha+R} + |(q)_{R+1}| \left(\frac{1}{\sigma+R} - \frac{1}{\alpha+R} \right) = \mathcal{A}_R.$$

The remaining scaling factor is $K^{-\alpha-R}$, proving (28) for every complex phase. □

Corollary 7.2 (Sharp fixed-trial alignment)**.** *For a fixed nonzero anchored, q-null trial let*

$$\mathcal{B}_c(K) = \sum_d c_d d B_2(\{K/d\}), \qquad W(c) = \operatorname{av}_K |\mathcal{B}_c(K)|^2.$$

The average is over a common integer period. Then

$$m - m_M = a\mathcal{B}_c(K) K^{-\lambda-1} + O_{a,q,c}(K^{-\alpha-2}), \tag{30}$$

$$W(c) = \frac{1}{180} \sum_k (J_4(k) + 10 J_2(k)) \left| \sum_{k|d} \frac{c_d}{d} \right|^2 > 0, \tag{31}$$

$$\limsup_{M\to\infty} K^{\alpha+1} |m - m_M| = a \max_K |\mathcal{B}_c(K)| > 0. \tag{32}$$

Writing $Q_{\mathrm{tail}} = Q - Q_M$ and $\pi_M = K^{-2a}$, one also has

$$\frac{K^2 |m - m_M|^2}{\pi_M Q_{\mathrm{tail}}} = \frac{a}{2V(c)} |\mathcal{B}_c(K)|^2 + O_{a,q,c}(K^{-1}). \tag{33}$$

Thus the actual tail alignment is $O(K^{-2})$, with the Cesàro mean of the scaled ratio tending to $aW(c)/(2V(c))$.

Proof. Use $R = 2$ in Theorem 7.1. Since $\Delta_1 = 2a$, the leading term is as stated; one explicit error constant in (30) is $C_2 S_2(|\Delta_2| + \mathcal{A}_2)$. For $g = \gcd(d, e)$ the exact discrete identity is

$$\operatorname{av}_n B_2(\{n/d\}) B_2(\{n/e\}) = \frac{g^4}{180 d^2 e^2} + \frac{g^2}{18 d^2 e^2} - \frac{1}{30 d^2 e^2}.$$

Indeed, conditional on $n \bmod g = z$, multiplication of Bernoulli polynomials gives means $(g/d)^2 B_2(z/g)$ and $(g/e)^2 B_2(z/g)$. Their conditional product is valid by the Chinese remainder theorem, and direct quartic summation gives $g^{-1} \sum_{z=0}^{g-1} B_2(z/g)^2 = 1/180 + 1/(18g^2) - 1/(30g^4)$. Anchoring cancels the last rank-one term and gives $\operatorname{av} \mathcal{B}_c = \sum c_d/(6d) = 0$. The divisor identities for J_4, J_2 prove (31). The largest occupied divisor coordinate proves its strict positivity.

This covariance is credited to [11, Theorem 1]: its wave is $b_d(k) = d B_2(\{k/d\})/2 - 1/(12d)$ and its correlation is $(g^2-1)(g^2+11)/(720de)$. On the anchored space $\mathcal{B}_c = 2\sum c_d b_d$, so (31) is four times that correlation form. The calculation here fixes its normalization for the matched Poisson mean.

Periodicity and the uniform-in-residue error in (30) prove (32); the corresponding Cesàro squared limit is $a^2 W(c)$. Finally Theorem E.1 gives $Q_{\mathrm{tail}} = 2aV(c)K^{-1}(1 + O(K^{-1}))$, proving (33). All these asymptotics hold with the trial, height and parameters fixed; their constants need not be uniform when support or coefficient masses grow. □

8 A complete finite model for optimized gain

The full moving-support norm estimate and compensated mean can be combined before optimization. This controls approximation of the actual optimal gain, with a target implication that prices the complete approximation error.

Fix $0 < a < 1/2$, $\sigma = 1/2 - a$, $\alpha = 1/2 + a$, $q = \sigma + iT$, $\lambda = \alpha + iT$, and let

$$\mathcal{H}_N = \{c \in \mathbb{C}^N : A_c(1) = A_c(q) = 0\}.$$

All suprema below use this same exact double-null space, assumed nontrivial. They need not equal an optimum on the larger space with only $A(1) = 0$. Retain

$$V(c) = \sum_{k=2}^{N} \frac{J_2(k)}{12} |t_k|^2, \qquad C_N = 12 \sum_{k=2}^{N} \frac{\sigma_1(k)^2}{J_2(k)}, \qquad S_0^2 \le C_N V(c).$$

Suppose the complete cotangent model is

$$\widehat{Q}(c) = Q_M(c) + \frac{2a}{K} W(c), \qquad W(c) \ge V(c), \qquad K = M + 1,$$

and its proved finite error satisfies

$$|Q(c) - \widehat{Q}(c)| \le \frac{2a}{K} H V(c) \le H \widehat{Q}(c), \qquad 0 \le H < 1. \tag{34}$$

In particular the complete common head Q_M is retained.

Theorem 8.1 (Corrected optimized gain)**.** *Let $R \ge 1$, $L_q = |q| + R + 1$, and suppose $0 < \theta < 1$ and $K \ge N L_q/(2\pi\theta)$. Use the actual corrected row $b = b_M + t_{R,K}$ from the compensated mean formula, and define*

$$e = \frac{2\sqrt{2a}}{\pi} \sqrt{C_N}\, K^{-a} \theta^R \left(2 + \frac{L_q}{R^2}\right),$$

$$G_b = \sup_{c \in \mathcal{H}_N \setminus \{0\}} \frac{|bc|^2}{\widehat{Q}(c)}, \qquad G_{\text{true}} = \sup_{c \in \mathcal{H}_N \setminus \{0\}} \frac{|m(c)|^2}{Q(c)}.$$

Then

$$\frac{[\sqrt{G_b} - e]_+^2}{1 + H} \le G_{\text{true}} \le \frac{[\sqrt{G_b} + e]^2}{1 - H}. \tag{35}$$

There is also the explicit additive bound

$$|G_{\text{true}} - G_b| \le H + 2e\sqrt{1 + H} + e^2. \tag{36}$$

In particular

$$G_b > \left[\sqrt{\eta(1 + H)} + e\right]^2 \quad \Longrightarrow \quad G_{\text{true}} > \eta. \tag{37}$$

Proof. The complete compensated-mean estimate is

$$|m - bc| \le C_R S_R K^{-\alpha - R} \mathcal{A}_R,$$

$$\mathcal{A}_R = \frac{|(\lambda)_{R+1} - (q)_{R+1}|}{\alpha + R} + \frac{2a|(q)_{R+1}|}{(\sigma + R)(\alpha + R)}.$$

The Bernoulli Fourier estimate gives $C_R \le 4/(2\pi)^{R+1}$, and $S_R \le N^R\sqrt{C_N V(c)}$. Along the segment from q to λ, each factor in the rising product of length $R+1$ has magnitude at most L_q, since $2a < 1$. Differentiate that product and integrate over the segment to obtain

$$|(\lambda)_{R+1} - (q)_{R+1}| \le 2a(R+1)L_q^R, \qquad |(q)_{R+1}| \le L_q^{R+1}.$$

Because $(R+1)/(R+\alpha) \le 2$ and $(R+\sigma)(R+\alpha) \ge R^2$, it follows that

$$\mathcal{A}_R \le 2aL_q^R(2 + L_q/R^2).$$

Use $NL_q/(2\pi K) \le \theta$ and $\widehat{Q} \ge 2aV/K$ to conclude

$$|m - bc| \le e\sqrt{\widehat{Q}(c)}. \tag{38}$$

The divisor transform is invertible on the anchored finite space, so $V(c) > 0$ for nonzero $c \in \mathcal{H}_N$. Thus $\widehat{Q}$ is a norm square there. Let

$$E = \sup_{c \in \mathcal{H}_N \setminus \{0\}} |m(c)|^2/\widehat{Q}(c).$$

The triangle inequality for dual norms in (38) gives $|\sqrt{E} - \sqrt{G_b}| \le e$, including vanishing rows. The complete norm comparison (34) gives

$$E/(1+H) \le G_{\text{true}} \le E/(1-H),$$

proving (35). Also $|E - G_{\text{true}}| \le HG_{\text{true}} \le H$ and $E \le (1+H)G_{\text{true}} \le 1 + H$, since true native gain is at most1. Consequently

$$|G_b - E| \le 2e\sqrt{E} + e^2 \le 2e\sqrt{1+H} + e^2,$$

which proves (36); (37) follows from the lower bound. All coefficient constraints, head terms and errors are on the same space before taking suprema. □

For an injective exact-null basis $c = Jx$,

$$G_b = bJ(J^*\widehat{Q}J)^{-1}J^*b^*.$$

Every matrix and error uses this same congruence. This exact formula does not by itself certify a numerical inverse.

Fix a and $\gamma > 1/(2\pi)$, and choose

$$\theta = \frac{1 + 1/(2\pi\gamma)}{2}, \qquad K = \lceil \gamma NT \rceil, \qquad R = \max\left(1, \left\lceil \frac{3\log T}{\log(1/\theta)} \right\rceil\right).$$

The wall is eventually admitted uniformly in N. The complete Farey norm comparison has $H = O_{a,\gamma}(N/T + 1/T)$, while $C_N = O(N)$ and the preceding theorem give, for $2 \le N \le T$,

$$e = O_{a,\gamma}(N^{1/2-a}T^{-a-2}) = O_{a,\gamma}(T^{-3/2-2a}).$$

Hence along $N = o(T)$ the corrected finite model approximates the actual optimized double-null gain uniformly in an additive sense. The finite target condition (37) converts a certified model gain into a true gain at each admitted height. Its error is additive, so its quantitative payment remains explicit when the model gain is small.

Corollary 8.2 (Loss of a model-selected trial)**.** *Under Theorem 8.1, put* $\varepsilon = H + 2e\sqrt{1+H} + e^2$. *On any nontrivial exact support and null subspace, let* c_b *maximize* $|bc|^2/\widehat{Q}(c)$. *Then*

$$0 \le G_{\text{true}} - g(A_{c_b}) \le 2\varepsilon.$$

If a computed nonzero trial c *has model value at least* $G_b - \delta$, $\delta \ge 0$, *its loss is at most* $2\varepsilon + \delta$.

Proof. Apply the additive bound first to the whole coefficient subspace and then to the one-dimensional span of c. The latter gives $|g(A_c) - |bc|^2/\widehat{Q}(c)| \le \varepsilon$ with the same constants. Thus $G_{\text{true}} - g(A_c) \le |G_{\text{true}} - G_b| + (G_b - |bc|^2/\widehat{Q}(c)) + \varepsilon$. The model maximum is attained on its finite-dimensional unit sphere because $\widehat{Q}$ is positive definite. □

9 Projection disks and arithmetic searches

For completeness, let $\Omega = \{\Re s > 1/2\}$ and $k_\lambda(s) = (s + \overline{\lambda} - 1)^{-1}$ be its Hardy kernel. The image of A is $f_A(s) = a\zeta(s)A(s)/(s-q)$, with $\|f_A\|_{H^2}^2 = aQ(A)/2$. The normalized squared projection of ak_λ onto the span of these images is g_N. If $\zeta(\rho) = 0$, $\rho \in \Omega$, every image vanishes at ρ. Projecting the kernel off k_ρ therefore gives

$$g_N(\lambda) \le \left| \frac{\rho - \lambda}{\rho + \overline{\lambda} - 1} \right|^2.$$

This is the classical local projection mechanism of [1]. A proved lower certificate $L > 0$ thus excludes zeros from the disk

$$\left| \frac{s - \lambda}{s + \overline{\lambda} - 1} \right| < \sqrt{L}.$$

An implementation must use enclosures for the finite sums in (62) or (74), and pay the coefficient growth in (59) and the explicit correction costs. The upper dual in (51) and the lower certificate measure opposite sides of the same complete arithmetic problem. A family whose certified lower gains exceed a prescribed geometric threshold throughout a height range supplies zero-free disks throughout that range. The compensated numerator retains the exact finite-trial cancellation.

9.1 Continuous height coverage from one seed

Fix the same anchored A, a seed T_0, and $0 \le L_0 \le g_{T_0}$. Set

$$x = |T - T_0|/a, \qquad C(x) = \frac{\sqrt{4 + x^2} + x}{2}.$$

Theorem 9.1 (Complete-norm height continuation)**.** *For every real* T,

$$g_T \ge \left[\frac{\max\{0, \sqrt{L_0} - (x/2)\sqrt{1 - L_0}\}}{C(x)} \right]^2. \tag{39}$$

For a target $\eta < L_0$, *the same trial supplies* $g_T \ge \eta$ *throughout*

$$|T - T_0| \le a\, \frac{\sqrt{L_0} - \sqrt{\eta}}{\frac{1}{2}\sqrt{1 - L_0} + \sqrt{\eta}}. \tag{40}$$

Proof. Let $F_0(s) = a\zeta(s)A(s)/(s - q_{T_0})$. Proposition 1.1 gives $\|F_0\|^2 = aQ_0/2$ and $F_0(\lambda_{T_0}) = m_0/2$. Its orthogonal decomposition along the Hardy kernel is

$$F_0 = am_0 k_{\lambda_{T_0}} + R_0, \qquad \|R_0\|^2 = (aQ_0/2)(1 - g_0), \quad R_0(\lambda_{T_0}) = 0.$$

Writing $\Delta = T - T_0$, direct kernel algebra gives

$$\|k_{\lambda_T} - P_{k_{\lambda_{T_0}}} k_{\lambda_T}\|^2 = \frac{\Delta^2}{2a(4a^2 + \Delta^2)}.$$

Since $m_T = (2a+i\Delta)F_0(\lambda_T)/a$, its projected part is exactly m_0. Thus $|m_T - m_0| \le \sqrt{Q_0(1 - g_0)}\, x/2$, preserving its complex phase. The boundary multiplier from F_0 to F_T has squared supremum

$$\sup_{u\in\mathbb{R}} \frac{a^2 + (u + \Delta)^2}{a^2 + u^2} = C(x)^2,$$

so $Q_T \le C(x)^2 Q_0$. The function $\sqrt{g} - (x/2)\sqrt{1 - g}$ is increasing on $[0, 1]$; replace g_0 by its lower L_0 to prove (39). Finally $C(x) \le 1 + x$ gives (40). □

The exterior null is imposed at the seed to certify its arithmetic tails. Height continuation uses the same anchored trial and its complete Hardy norm; it does not impose that extra null again at neighboring heights. A finite union of the certified intervals gives continuous compact-range coverage.

10 A degree-uniform obstruction for geometric modules

The complete receiver permits growing arithmetic support. Increasing the degree within a fixed geometric dictionary, however, cannot by itself produce a positive all-height gain. The obstruction below uses the original complete norm, rather than a particular finite certificate.

Keep $0 < a < 1/2$, $\alpha = 1/2 + a$, $\lambda = \alpha + iT$ and $q = 1/2 - a + iT$. For every nonzero finite anchored polynomial $A(1) = 0$, put

$$F(s) = \frac{a\zeta(s)A(s)}{s - q}.$$

The complete recipient gives $F \in H^2(\Re s > 1/2)$, $\|F\|^2 = aQ(A)/2$ and $F(\lambda) = m(A)/2$. The Hardy norm uses $(2\pi)^{-1}\int_{\mathbb{R}}$ on the boundary. The additional null $A(q) = 0$ is not required in this section; every conclusion remains valid on that smaller coefficient space.

Theorem 10.1 (Single geometric lattice)**.** *Fix an integer $b \ge 2$ and let $h = 2\pi/\log b$. Suppose the support of A consists of powers of b. Write $Z_0 = \zeta(\lambda)$ and $Z_1 = \zeta(\lambda + ih)$. If $Z_0 \ne 0$, then*

$$g(A) \le \frac{h^2|Z_0|^2}{h^2|Z_0|^2 + 4a^2|Z_1 - Z_0|^2}. \tag{41}$$

If $Z_0 = 0$, then $g(A) = 0$. The bound is independent of degree and coefficient magnitudes.

Proof. Here $A(s + ih) = A(s)$ exactly. The Hardy kernel is $k_w(s) = (s + \overline{w} - 1)^{-1}$. For $w = \lambda + ih$, the squared norm of the evaluation residual after projection onto k_λ is $h^2/[2a(4a^2 + h^2)]$. With $\kappa = 2a/(2a + ih)$, its value on F is

$$F(w) - \kappa F(\lambda) = \frac{aA(\lambda)(Z_1 - Z_0)}{2a + ih}.$$

Orthogonal projection onto the two kernels therefore gives

$$\|F\|^2 \geq 2a|F(\lambda)|^2 + \frac{|F(w) - \kappa F(\lambda)|^2}{h^2/[2a(4a^2+h^2)]}.$$

Substitution yields

$$Q(A) \geq |A(\lambda)|^2 \left(|Z_0|^2 + \frac{4a^2}{h^2}|Z_1 - Z_0|^2 \right).$$

If $A(\lambda) = 0$ the gain vanishes; otherwise division proves (41). The kernel inequality is sharp in the ambient Hardy space; arithmetic attainment is not asserted. □

Fix b and positive integers $d_1, \ldots, d_m$. A geometric module here means the polynomials

$$A(s) = \sum_{r=1}^{m} d_r^{-s} P_r(b^{-s}), \tag{42}$$

where each P_r is any finite polynomial. Both its coefficients and degree may depend on T. Put $z_r = \exp(-ih \log d_r)$ and define

$$1 - \prod_{r=1}^{m}(1 - z/z_r) = \sum_{j=1}^{m} \ell_j z^j, \qquad L = \sum_{j=1}^{m} |\ell_j| \sqrt{1 + (jh/(2a))^2}.$$

Repeated nodes are allowed. Elementary symmetric functions give

$$|\ell_j| \leq \binom{m}{j}, \qquad L \leq (2^m - 1)\sqrt{1 + (mh/(2a))^2}. \tag{43}$$

Theorem 10.2 (Fixed modules and finite menus)**.** *For A as in (42), let $Z_j = \zeta(\lambda + ijh)$ and assume $Z_j \neq 0$ for $1 \leq j \leq m$. Every nonzero anchored A satisfies*

$$g(A) \leq |Z_0|^2 \left(\sum_{j=1}^{m} \frac{|\ell_j| \sqrt{1 + (jh/(2a))^2}}{|Z_j|} \right)^2 \leq \frac{L^2 |Z_0|^2}{\min_{1 \leq j \leq m} |Z_j|^2}. \tag{44}$$

Let $G_{b,\boldsymbol{d}}(T)$ be the supremum of true gain over all nonzero anchored polynomials in this module, with no degree bound. For every $\eta > 0$, the set of $T \geq 0$ for which $G_{b,\boldsymbol{d}}(T) < \eta$ has positive lower density. The same assertion holds for the maximum over any finite predeclared collection of fixed modules, including different bases. In particular each such all-degree optimum has lower limit zero.

Proof. Set $A_j = A(\lambda + ijh)$. The defining polynomial takes value1 at each z_r, so the exact shift relation is

$$A_0 = \sum_{j=1}^{m} \ell_j A_j.$$

Complete Hardy evaluation at each shifted point gives

$$Q(A) \geq \frac{4a^2}{4a^2 + (jh)^2} |Z_j A_j|^2.$$

The triangle inequality proves (44).

Use the standard Voronin–Reich–Bagchi universality theorem in its compact-set form: non-vanishing continuous targets, analytic in the interior of a compact subset of $1/2 < \Re s < 1$

with connected complement, are approximated within any fixed positive tolerance by a positive-lower-density set of vertical shifts; see [9, p. 59]. Choose small disjoint closed disks centered at $\alpha, \alpha + ih, \dots, \alpha + imh$, contained in that strip. Their union has connected complement. Take target $\varepsilon > 0$ on the central disk and target1 on each other disk. These targets are nowhere zero. Approximation with tolerance $\varepsilon < 1/2$ gives

$$|Z_0| < 2\varepsilon, \qquad |Z_j| > 1 - \varepsilon > 1/2 \quad (1 \le j \le m)$$

on a positive-lower-density set of actual T. Hence the supremum in (44) is less than $16L^2\varepsilon^2$. Choose also $\varepsilon < \sqrt{\eta}/(4L)$. For a finite menu, use the finite union of all distinct positive probe offsets and the maximum of the corresponding constants L; coincident probes receive the same target1. This is one universality application on a fixed compact set. Suprema are measurable: fixed anchored finite coefficient spaces admit countable dense sets of trials with continuous gain, and only countably many finite degrees are involved. □

Proposition 10.3 (A paid escape condition)**.** *Keep the fixed nodes and assume* $\min_{1\le j\le m} |Z_j| > 0$. *For any finite anchored arithmetic polynomial, not necessarily supported in the module, set*

$$J(z) = \prod_{r=1}^{m} (1 - z/z_r), \qquad E(A) = \sum_d c_d d^{-\lambda} J(e^{-ih \log d}).$$

Then

$$A_0 - E(A) = \sum_{j=1}^{m} \ell_j A_j, \qquad |A_0 - E(A)| \le \frac{L\sqrt{Q(A)}}{\min_{1\le j\le m} |Z_j|}. \tag{45}$$

Any family with $|E(A)| \le \delta |A_0|$ *for a fixed* $0 \le \delta < 1$ *satisfies* (44) *with an additional factor* $(1-\delta)^{-2}$, *and therefore has the same all-height obstruction. Conversely, if* $g(A) \ge \eta > 0$ *and* $Z_0 \ne 0$, *then*

$$\left| 1 - \frac{E(A)}{A_0} \right| \le \frac{L|Z_0|}{\sqrt{\eta} \min_{1\le j\le m} |Z_j|}. \tag{46}$$

If off-module terms denote those indices outside $\bigcup_{r=1}^{m} \{d_r b^k : k \ge 0\}$, *their coefficient cost satisfies*

$$\sum_{d \ \mathit{off\ module}} |c_d| d^{-\alpha} \ge 2^{-m} |A_0| \left[1 - \frac{L|Z_0|}{\sqrt{\eta} \min_{1\le j\le m} |Z_j|} \right]_+ .$$

Proof. The polynomial identity defining J gives the first part of (45) term by term. The same complete evaluations used above give its inequality. The relative-defect conclusion follows from $|A_0 - E(A)| \ge (1-\delta)|A_0|$. For a successful trial, $A_0 \ne 0$ and $\sqrt{Q}/|A_0| = |Z_0|/\sqrt{g(A)}$; division proves (46). Module-supported terms contribute zero to E, and $|J(z)| \le 2^m$ on the unit circle. These facts and the reverse triangle inequality prove the last bound. Exact module support implies $E = 0$; the converse is not claimed. □

The conclusions hold for every geometric depth with the base and finite module generators fixed. Their positive-lower-density assertion is qualitative. On any support schedule satisfying Theorem 1.2, the additive corrected-model comparison transfers the lower-limit-zero conclusion to the *same restricted module subspace*. The paid escape condition quantifies the off-module evaluation required of a successful growing dictionary construction.

10.1 Joint probes and the complex escape defect

The preceding bound treats the shifted evaluations separately. Their joint Hardy Gram matrix gives a sharper bound and records the phase of the required escape. This is finite reproducing-kernel projection, applied to the arithmetic shift relation above.

Proposition 10.4 (Joint-probe bound)**.** *Keep the fixed module data and suppose $Z_j \neq 0$ for $0 \le j \le m$. For $0 \le i, j \le m$ define*

$$K_{ij} = \frac{1}{2a + \mathrm{i}(i-j)h}, \qquad \gamma_j = \frac{Z_0 \ell_j (2a + \mathrm{i}jh)}{2aZ_j} \quad (1 \le j \le m), \qquad w = (1, -\gamma_1, \dots, -\gamma_m).$$

Let $e_0 = (1, 0, \dots, 0)$ be a row, and set

$$u = \frac{1}{2a}, \quad b = e_0 K w^*, \quad d = wKw^*, \quad D = ud - |b|^2 > 0.$$

For any nonzero anchored arithmetic trial with $A_0 \neq 0$, put $r = E(A)/A_0$. Then

$$g(A) \le \frac{2aD}{d - 2\Re(br) + u|r|^2}. \tag{47}$$

For module-supported trials this gives

$$g(A) \le 1 - \frac{2a|b|^2}{d}.$$

Every trial with $g(A) \ge \eta$, $0 < \eta \le 1$, satisfies

$$\left| \frac{E(A)}{A_0} - 2a\bar{b} \right| \le 2a\sqrt{D(\eta^{-1} - 1)}. \tag{48}$$

If $A_0 = 0$ or $Z_0 = 0$, the gain is zero by the mean identity.

Proof. The distinct evaluation points $\lambda + \mathrm{i}jh$ have positive definite Hardy Gram matrix K. The row w is independent of e_0, because at least one ℓ_j is nonzero. Thus the Gram matrix of the center evaluation and their linear combination is

$$B = \begin{pmatrix} u & b \\ \bar{b} & d \end{pmatrix} > 0.$$

For $F(s) = a\zeta(s)A(s)/(s-q)$, let $f_j = F(\lambda + \mathrm{i}jh)$. The exact defect identity gives $wf = Z_0 E(A)/2 = rf_0$, while $f_0 = m/2$. Projection onto the two representers therefore yields

$$\frac{aQ}{2} = \|F\|^2 \ge \frac{|m|^2}{4D}\big[d - 2\Re(br) + u|r|^2\big].$$

Division proves (47); its denominator is positive because $B > 0$. Module support gives $r = 0$. For $g \ge \eta$, complete the square in the denominator using $u = 1/(2a)$ to obtain (48). □

The module bound is the exact center-evaluation bound on the kernel of the joint shift functional in the ambient Hardy space. It is therefore at least as strong as discarding correlations and using the separate-probe triangle bound. For one lattice it reduces to Theorem 10.1. With fixed probes bounded away from zero and $Z_0 \to 0$, one has $b \to u$ and $D \to 0$; hence every fixed positive-gain trial must have $E(A)/A_0 \to 1$, including its phase.

11 Numerical examples with complete tail bounds

All three cases use $a = 18667/50000$, $\alpha = 43667/50000$, $\sigma = 6333/50000$. The public verifier uses exact rational Bernoulli values, reduced-divisor covariance and discrete seam identities. Directed 512-bit Arb/Acb calculations then test the actual complete arithmetic head and both infinite mean/norm tails. Its three-coefficient family is defined analytically by

$$c_3 = 1, \quad c_2 = \frac{3^{-q} - 1/3}{1/2 - 2^{-q}}, \quad c_1 = -c_2/2 - 1/3.$$

Both $A(1) = A(q) = 0$ hold identically. The denominator cannot vanish because $|2^{-q}| = 2^{-\sigma} > 1/2$. Each head is accumulated from its same actual B_n states; the new covariance weights use the same complex coefficients and full mixed orders.

The mean uses Theorem 7.1 at order $R+2$. The accepting numerator is its finite corrected row minus its complete error. The denominator is $Q_M + U_R(c) + \delta_R S_0^2$. Direct evaluation of $\zeta(\lambda)A(\lambda)$ is an independent remainder control. The resulting rounded-down gains and height radii are

T	K	R	θ	complete gain lower	height halfwidth
1000	1024	23	1/2	.1971551245	.3685275677
10000	16384	30	1/2	.03521509868	.14171006664
1000	768	56	3/4	.1969254372	.3682598316

The halfwidths use (40) for the illustrative target gain $\eta = 81/10^8$. The first two complete norm-tail uppers are, respectively, less than .000176596358 and .000010561003. Compared with the scalar $25S_0^2/(16K)$ envelope, their tail tightenings exceed 42.1372 and 92.7486. The third uses the general $\theta = 3/4$ wall, below the half-ratio wall; its comparison is the independently paid $(b_\theta + e_\theta)^2 S_0^2/K$ envelope, with tightening greater than 42.1073. These are bounds on tail certification; the head and corrected mean remain common in each comparison.

The accompanying programs check the whole-period mixed source against its divisor form, the Bernoulli covariance identities, the higher-Jordan weights and independently summed finite tail segments. These finite controls supplement the analytic proofs; the reader instructions give the dependencies and reproduction commands.

A Exact evaluation duals

Finite head inversion and harmonic–Möbius duality have close antecedents in [15, 16]. The following formulas retain the present shifted evaluation and complete arithmetic recipient.

Write μ for the Möbius function and

$$M_x(z) = \sum_{m \le x} \mu(m) m^{-z}.$$

For $1 \le n \le N$ set

$$v_n = n^{-2a} M_{N/n}(\lambda), \qquad v_{N+1} = 0, \quad \ell_n = v_n - v_{n+1}, \quad \mathcal{D}_N = \sum_{n \le N} \frac{|\ell_n|^2}{p_n}. \tag{49}$$

Theorem A.1 (Evaluation-first dual)**.** *For every coefficient vector supported at most N,*

$$A(\lambda) = \sum_{n \le N} \ell_n B_n.$$

The squared norm of this functional on the unrestricted first-head space with norm $\sum_{n\le N} p_n|B_n|^2$ *is exactly* $\mathcal{D}_N$*. For the anchored complete recipient,*

$$g_N(\lambda) \le \min\{1, |\zeta(\lambda)|^2 \mathcal{D}_N\}. \tag{50}$$

Proof. Finite inversion gives

$$c_k = \sum_{d|k} \mu(k/d) d^q (B_d - B_{d-1}), \qquad B_0 = 0.$$

Substitution in $\sum c_k k^{-\lambda}$ yields $\sum_{d\le N} d^{q-\lambda} M_{N/d}(\lambda)(B_d - B_{d-1})$. Since $q - \lambda = -2a$, summation by parts gives the stated row. Weighted Cauchy–Schwarz proves the norm bound, with equality in the unrestricted head at $B_n = \overline{\ell_n}/p_n$. The full Q is at least its positive first head, so (3) proves (50). □

Anchoring can be retained exactly, rather than discarded in this upper bound. Put

$$w_n = n^{q-1} M_{N/n}(1), \quad w_{N+1} = 0, \quad h_n = w_n - w_{n+1}.$$

The same inversion gives $A(1) = \sum h_n B_n$. In particular $h \neq 0$, since $h_N = N^{q-1}$. The exact squared head norm restricted to anchoring is

$$\mathcal{D}_N^{\mathrm{anch}} = \mathcal{D}_N - \frac{\left|\sum_{n\le N} \ell_n \overline{h_n}/p_n\right|^2}{\sum_{n\le N} |h_n|^2/p_n}. \tag{51}$$

This follows by orthogonal projection after setting $x_n = \sqrt{p_n} B_n$. Thus $\mathcal{D}_N$ in (50) may be replaced by $\mathcal{D}_N^{\mathrm{anch}}$. Neither head norm is asserted equal to the optimized gain in the complete arithmetic recipient.

B A log-free power law and a necessary cutoff cost

Let $\delta = 1 - \alpha = \sigma$ and $b = 1 - 2a = 2\delta$. In the following lemma K, L are dummy integer cutoffs. For $K \ge 1$ define a finite sequence $H_K(n) = n^{-2a} \mathbf{1}_{n\le K}$ and its forward difference $\Delta H_K(n) = H_K(n) - H_K(n+1)$.

Lemma B.1 (Rank-one plus diagonal Gram)**.** *In the difference norm with weights* $1/p_n$*,*

$$\langle \Delta H_K, \Delta H_L \rangle_{1/p} = 1 \quad (K \neq L), \qquad d_K := \|\Delta H_K\|_{1/p}^2 = 1 - K^{-2a} + \frac{K^{-4a}}{p_K}.$$

Writing

$$u_K = \sum_{\lfloor N/m \rfloor = K} \mu(m) m^{-\lambda},$$

one consequently has the exact identity

$$\mathcal{D}_N = |M_N(\lambda)|^2 + \sum_{K\le N} (d_K - 1)|u_K|^2. \tag{52}$$

Proof. The difference ΔH_K equals p_n for $n < K$, equals K^{-2a} at $n = K$, and vanishes after that. For $K < L$ the cross product is $\sum_{n<K} p_n + K^{-2a} = 1$. The diagonal formula follows similarly. Now $v = \sum_{m\le N} \mu(m) m^{-\lambda} H_{\lfloor N/m \rfloor}$. Expand its difference norm and group equal floor values. The all-ones part is $|\sum u_K|^2 = |M_N(\lambda)|^2$. □

Theorem B.2 (Uniform power law)**.** *For every $T \in \mathbb{R}$ and integer $N \geq 1$,*

$$N^b \left(\frac{1}{2a} - \frac{1}{N} \right) \leq \mathcal{D}_N \leq K_a N^b, \tag{53}$$

where one admissible explicit constant is

$$K_a = \delta^{-2} + \frac{2^{2a}}{a}(2^{2\alpha+2} + 1)\zeta(2).$$

In particular $g_N(\lambda) \leq K_a N^{2-2\alpha}|\zeta(\lambda)|^2$. The exponent in (53) *is sharp for $\mathcal{D}_N$.*

Proof. The integral formula for p_K gives

$$a2^{-2a}K^{-2a-1} \leq p_K \leq 2aK^{-2a-1}, \qquad 0 < d_K - 1 \leq \frac{2^{2a}}{a}K^b.$$

Also $|M_N(\lambda)| \leq \sum_{m \leq N} m^{-\alpha} \leq N^\delta/\delta$. If $K \leq \sqrt{N}$, the interval $N/(K+1) < m \leq N/K$ contains at most $2N/K^2$ integers. Hence

$$|u_K| \leq 2^{\alpha+1}N^\delta K^{\alpha-2}, \qquad K^b|u_K|^2 \leq 2^{2\alpha+2}N^b/K^2.$$

If $K > \sqrt{N}$, that interval has length less than one. Each nonempty group contains just one m, and its contribution is at most N^b/m^2. Summing both regimes proves the upper bound via (52). This is the weighted quotient-increment split underlying the related harmonic estimates of [17].

All terms in (52) are nonnegative. The group $K = N$ contains only $m = 1$, so $u_N = 1$ and

$$\mathcal{D}_N \geq d_N - 1 = \frac{N^{-4a}}{p_N} - N^{-2a} \geq N^b \left(\frac{1}{2a} - \frac{1}{N} \right).$$

This sharpness statement concerns the unrestricted head functional; anchoring and the full tail remain present in the gain inequality. □

Corollary B.3 (All-height gain requires growing support)**.** *Fix $\alpha \in (1/2, 1)$ and $\eta > 0$. If an arbitrary integer schedule $N(t) \geq 2$ satisfies $g_{N(t)}(\alpha + it) \geq \eta$ for all sufficiently large t, then along an unbounded sequence t_j,*

$$\log N(t_j) \geq \frac{\sqrt{\alpha - 1/2}}{20(1-\alpha)} \frac{(\log t_j)^{1-\alpha}}{(\log\log t_j)^\alpha} + O_{\alpha,\eta}(1). \tag{54}$$

Thus every cutoff with logarithm of smaller order than the displayed height scale fails a fixed positive gain requirement infinitely often.

Proof. Montgomery's unconditional Theorem 1 [18], at angle π, gives for every sufficiently large X a height $X^{(\alpha-1/2)/3} \leq t \leq X$ such that

$$|\zeta(\alpha + it)| \leq \exp\left[-\frac{\sqrt{\alpha - 1/2}}{20} \frac{(\log X)^{1-\alpha}}{(\log\log X)^\alpha} \right].$$

The scale is eventually increasing, so it can be replaced by its value at t. These heights are unbounded. Apply Theorem B.2 and take logarithms of $\eta \leq K_a N(t)^b |\zeta(\alpha+it)|^2$. Since $b = 2(1-\alpha)$, this is (54). □

The source's sufficiently-large onset is not replaced by a numerical height here. The statement fixes α, permits all complex coefficient choices at every height, and concerns the largest permitted integer index rather than coefficient cardinality.

C Null adaptation without loss of the closed trial space

The support lower cost does not construct a successful trial. We next give a complete way to certify any proposed growing trial.

Theorem C.1 (Costed exterior-null adaptation)**.** *For nonzero $A \in \mathcal{A}_N$ and integer $k \geq 1$, set*

$$\varepsilon = 2^{-ka}, \qquad A_k(s) = A(s)(1 - 2^{k(q-s)}).$$

Then $A_k(1) = A_k(q) = 0$ and its support is at most $2^k N$. Moreover

$$Q(A_k - A) = \varepsilon^2 Q(A), \tag{55}$$

$$(1-\varepsilon)^2 Q(A) \leq Q(A_k) \leq (1+\varepsilon)^2 Q(A), \tag{56}$$

$$m(A_k) = (1-\varepsilon^2) m(A), \tag{57}$$

$$(1-\varepsilon)^2 g(A) \leq g(A_k) \leq (1+\varepsilon)^2 g(A). \tag{58}$$

The arithmetic images satisfying both nulls have the same closed span as all anchored arithmetic images, at every fixed a, T.

Proof. On $\Re s = 1/2$, the multiplier $2^{k(q-s)}$ has constant modulus ε. Equation (4) proves (55); the triangle inequality proves the two norm bounds. At λ its value is $2^{-2ka} = \varepsilon^2$, proving the mean identity and the gain bounds. The two nulls and support claim are immediate, with any coincident integer coefficients added. As $k \to \infty$, the full-norm error tends to zero, proving equality of closures. The reverse inclusion holds because the doubly null polynomials are already anchored. □

The transformed coefficient mass obeys

$$S_0(A_k) \leq (1 + 2^{k\sigma}) S_0(A). \tag{59}$$

Thus small gain loss is accompanied by an explicit cost both in support and in coefficient mass; the construction is not cost-free as $a \to 0$.

D Complete finite arithmetic lower certificates

Theorem D.1 (Tail-paid lower gain)**.** *Suppose $A \neq 0$ and $A(1) = A(q) = 0$, and put*

$$\mathcal{K} = \left(1 + \frac{|q|}{\sigma}\right) S_0(A), \qquad m_M = \sum_{n \leq M} p_n B_n, \quad Q_M = \sum_{n \leq M} p_n |B_n|^2.$$

Then for every integer $M \geq 1$,

$$0 \leq Q - Q_M \leq \frac{\mathcal{K}^2}{M+1}, \tag{60}$$

$$|m - m_M| \leq \frac{\mathcal{K}}{(M+1)^\alpha}. \tag{61}$$

Consequently the finite arithmetic quantity

$$L_M(A) = \frac{\left(\max\{0, |m_M| - \mathcal{K}/(M+1)^\alpha\}\right)^2}{Q_M + \mathcal{K}^2/(M+1)} \tag{62}$$

satisfies $0 \leq L_M(A) \leq g(A)$ and tends to $g(A)$ as $M \to \infty$. No zeta value is needed to evaluate (62).

Proof. Partial summation and boundedness of D give

$$\lim_{n\to\infty} B_n = q\int_1^\infty D(x)x^{-q-1}\,\mathrm{d}x = \zeta(q)A(q) = 0.$$

Subtracting this limit yields

$$B_n = D(n)n^{-q} - q\int_n^\infty D(x)x^{-q-1}\,\mathrm{d}x, \qquad |B_n| \le \mathcal{K}n^{-\sigma}.$$

Since $\sum_{n>M} p_n = (M+1)^{-2a}$, positivity and $2a+2\sigma = 1$, $2a+\sigma = \alpha$ prove both tail bounds. The triangle inequality for m and upper bound for Q prove the certificate; convergence follows from the vanishing tails and $Q > 0$. □

Combining Theorems C.1 and D.1 gives a complete finite hierarchy at a fixed center. To approach the limiting optimized gain, first choose a finite anchored A near its supremum, then k to retain its gain, then M to approximate $g(A_k)$. For a monotone sequence of lower bounds, retain the largest certificate obtained so far. Completeness is an exhaustion statement; a successful uniform schedule must additionally bound its support, mass and tail parameters over the claimed height range.

As a literal example,

$$A(s) = (1-2^{1-s})(1-2^{q-s})$$

uses only $1, 2, 4$ and satisfies both nulls. Its center value is $(1-2^{1-\lambda})(1-2^{-2a}) \neq 0$ because $|2^{1-\lambda}| = 2^{1-\alpha} > 1$. It demonstrates the finite construction; its fixed cutoff remains subject to Corollary B.3.

E The sharp complete tail and a period-free remainder

The elementary tail upper in (60) can be refined to its leading coefficient. For a finite anchored coefficient vector put

$$V(c) = \frac{1}{12}\sum_{d,e} c_d\overline{c_e}\frac{\gcd(d,e)^2}{de}, \qquad S_1 = \sum_d d|c_d|, \qquad H = \frac{S_0}{2}. \tag{63}$$

Let $J_2(k) = k^2\prod_{p|k}(1-p^{-2})$ be the second Jordan function.

Theorem E.1 (Sharp null-adapted tail)**.** *For every nonzero finite A with $A(1) = A(q) = 0$, $V(c) > 0$. Set*

$$E = \frac{|q|S_1}{12}\left(1 + \frac{|q+1|}{\sigma+1}\right), \qquad C_0 = 2a(V(c)+S_1^2) + \frac{a(2a+1)H^2}{2}.$$

Then, for every integer $M \ge 1$,

$$\left|\sum_{n>M} p_n|B_n|^2 - \frac{2aV(c)}{M}\right| \le \frac{C_0}{M^2} + \frac{2HE}{(M+1)^2} + \frac{E^2}{(M+1)^3}. \tag{64}$$

In particular the M^{-1} order is exact for every fixed nonzero trial.

Proof. The continuous period mean of $D(x)$ is $\overline{D} = -\frac{1}{2}\sum c_d$. Define

$$h(x) = D(x) - \overline{D} = \sum_d c_d(1/2 - \{x/d\}), \qquad P(x) = -\frac{1}{2}\sum_d c_d d B_2(\{x/d\}),$$

where $B_2(u) = u^2 - u + 1/6$. The periodic continuous primitive P satisfies $P' = h$ almost everywhere, $|h| \le H$ and $|P| \le S_1/12$. The zero limit of B_n and integration by parts give

$$B_n = h(n)n^{-q} + qP(n)n^{-q-1} - q(q+1)\int_n^\infty P(x)x^{-q-2}\,\mathrm{d}x.$$

Thus $B_n = h(n)n^{-q} + R_n$ with

$$|R_n| \le En^{-\sigma-1}. \tag{65}$$

We require the discrete rather than continuous fractional-part covariance. Write $f_d(n) = 1/2 - \{n/d\}$ and $r = \gcd(d,e)$. Conditional on $n \bmod r = z$, the residues modulo d and e are independent on their admissible classes by the Chinese remainder theorem. Their conditional means are $(r/d)f_r(z)$ and $(r/e)f_r(z)$. Since $r^{-1}\sum_{z=0}^{r-1} f_r(z)^2 = 1/12 + 1/(6r^2)$, the exact discrete period mean is

$$\operatorname{av}_n f_d(n)f_e(n) = \frac{\gcd(d,e)^2}{12de} + \frac{1}{6de}. \tag{66}$$

The extra term cancels in the coefficient quadratic form by anchoring. Hence $\operatorname{av}|h(n)|^2 = V(c)$. The familiar continuous gcd covariance [2] is thus used here with its necessary discrete correction. The divisor identity for J_2 gives

$$12V(c) = \sum_k J_2(k)\left|\sum_{k\mid d}\frac{c_d}{d}\right|^2. \tag{67}$$

For a largest index with $c_d \neq 0$, the corresponding term is positive, which proves strict positivity.

For each pair (d,e), the centered sequence in (66) has mean zero, period $\operatorname{lcm}(d,e)$, and terms of modulus at most $1/2$. Therefore, writing $z_n = |h(n)|^2 - V(c)$,

$$\left|\sum_{n\le L} z_n\right| \le \frac{1}{2}\sum_{d,e}|c_d c_e|\operatorname{lcm}(d,e) \le \frac{S_1^2}{2} \qquad (L \ge 1).$$

Abel summation gives $|\sum_{n>M} z_n/n^2| \le S_1^2/(M+1)^2$. Also

$$|p_n - 2an^{-2a-1}| \le a(2a+1)n^{-2a-2}, \quad \frac{1}{M+1} \le \sum_{n>M} n^{-2} \le \frac{1}{M}.$$

It follows that

$$\left|\sum_{n>M} p_n|h(n)|^2 n^{-2\sigma} - \frac{2aV(c)}{M}\right| \le \frac{C_0}{M^2}.$$

Finally (65) bounds the difference of the squared terms by $2HEn^{-2\sigma-1} + E^2n^{-2\sigma-2}$. Summing against the full p_n and using their telescoping tail proves (64). No common-period length appears in its constants. □

The denominator of (62) may use the smaller of the proved upper tails $\mathcal{K}^2/(M+1)$ and $2aV/M$ plus the right-hand side of (64). Subtracting a known leading term can improve an enclosure, but additional zeros at q alone do not change the nonzero M^{-1} coefficient.

F Exact all-support tail capacity

The coefficient in Theorem E.1 itself has a sharp optimization independent of the full head norm. For complex z define $J_z(k) = \sum_{d|k} \mu(k/d) d^z$.

Theorem F.1 (Sharp tail-variance capacity)**.** *At every fixed center* (1)*, put*

$$S_\infty = \frac{\zeta(2)\zeta(2\alpha)}{|\zeta(1+\lambda)|^2} - 1 > 0.$$

Then

$$\inf_{\substack{A \text{ finite},\ A(1)=A(q)=0 \\ A(\lambda)=1}} V(c) = \frac{1}{12 S_\infty}. \tag{68}$$

Every zeta factor displayed in S_∞ is in its absolutely convergent half-plane. The infimum is approached by explicit finite polynomials.

Proof. Put $x_d = c_d/d$ and $t_k = \sum_{k|d} x_d$. On support at most N the inverse is $x_d = \sum_{m \le N/d} \mu(m) t_{dm}$, and

$$A(s) = \sum_{k \le N} t_k J_{1-s}(k), \qquad A(1) = t_1, \qquad 12V(c) = \sum_{k \le N} J_2(k) |t_k|^2. \tag{69}$$

Thus anchoring is $t_1 = 0$. On coordinates $2 \le k \le N$ define

$$u_k = \frac{J_{1-\lambda}(k)}{\sqrt{J_2(k)}}, \qquad v_k = \frac{J_{1-q}(k)}{\sqrt{J_2(k)}}.$$

The exact finite capacity is

$$S_N = \sum_{k=2}^{N} |u_k|^2 - \frac{|\sum_{k=2}^{N} u_k \overline{v_k}|^2}{\sum_{k=2}^{N} |v_k|^2}. \tag{70}$$

The denominator is nonzero for $N \ge 2$ since $2^{1-q} \ne 1$. When $S_N > 0$, the exact minimum of V under both nulls and $A(\lambda) = 1$ is $1/(12S_N)$, by projecting the evaluation row off the null row in the coordinates $y_k = \sqrt{J_2(k)} t_k$. Writing $C_N = \sum u_k \overline{v_k}$, an explicit minimizer is

$$y_k = \frac{\overline{u_k} - \overline{v_k}\, \overline{C_N} / \sum |v_j|^2}{S_N}.$$

It satisfies $\sum v_k y_k = 0$, $\sum u_k y_k = 1$, and $\sum |y_k|^2 = 1/S_N$. Formula (69) then constructs the coefficients. If $S_N = 0$, these normalized constraints are infeasible rather than an occasion to invert a singular matrix.

For $z = 1 - \lambda = \sigma - iT$, the prime-power formulas give

$$\begin{aligned} 1 + \sum_{j \ge 1} \frac{|J_z(p^j)|^2}{J_2(p^j)} &= 1 + \frac{|p^z - 1|^2}{(p^2-1)(1 - p^{2\sigma-2})} \\ &= \frac{|1 - p^{z-2}|^2}{(1-p^{-2})(1-p^{2\sigma-2})}. \end{aligned}$$

All factors converge absolutely since $2 - 2\sigma = 2\alpha > 1$ and $2 - \sigma > 1$. Multiplicativity and the positive series imply

$$\sum_{k \ge 1} \frac{|J_{1-\lambda}(k)|^2}{J_2(k)} = \frac{\zeta(2)\zeta(2\alpha)}{|\zeta(1+\lambda)|^2}. \tag{71}$$

In particular $u \in \ell^2$ with squared norm $S_\infty > 0$ after omitting coordinate 1. Cauchy–Schwarz in (69) already proves $V \geq |A(\lambda)|^2/(12S_\infty)$ for every finite anchored polynomial, even without the q-null condition.

In contrast, $v \notin \ell^2$. At primes,

$$\frac{|J_{1-q}(p)|^2}{J_2(p)} \geq (1-2^{-\alpha})^2 p^{2\alpha-2} \geq \frac{(1-2^{-\alpha})^2}{p},$$

and the reciprocal primes have divergent sum. Thus $\|v_{2:N}\| \to \infty$. The normalized rows converge weakly to zero: each fixed finite head divided by this norm tends to zero, and Cauchy–Schwarz bounds its pairing with the remaining part of u by that part's ℓ^2 norm. Consequently

$$\frac{|\sum_{k=2}^{N} u_k \overline{v_k}|}{\|v_{2:N}\|} \to 0, \qquad S_N \to S_\infty.$$

For all sufficiently large N, $S_N > 0$ and the finite minimizers therefore approach the lower bound, proving (68). □

Corollary F.2 (Uniform finite-cutoff capacity)**.** *Let*

$$L_\alpha = \frac{2-2^{-\alpha}}{(1-2^{-\alpha})^2}, \qquad \mathfrak{C}_\alpha = \exp(L_\alpha/\alpha).$$

For every real T and every integer $N \geq 2$ with $\log N \geq 1/(2a)$,

$$0 \leq S_\infty - S_N \leq N^{-2a}\left[e\zeta(2)(1+\log N)^4 + 4\mathfrak{C}_\alpha \zeta(2)^2(1+\log N)^8\right]. \tag{72}$$

Moreover $S_\infty \geq (2^\sigma - 1)^2/3$ uniformly in T. Thus a cutoff making the right side smaller than this positive constant guarantees feasibility of the normalized finite constraints for every T.

Proof. Write $\tau(n)$ for the divisor function and $H_N = \sum_{n \leq N} 1/n$. The elementary estimates $|J_z(n)| \leq n^{\Re z}\tau(n)$ and $J_2(n) \geq n^2/\zeta(2)$ imply

$$\left|\sum_{k=2}^{N} u_k \overline{v_k}\right| \leq \zeta(2) \sum_{k \leq N} \frac{\tau(k)^2}{k} \leq \zeta(2) H_N^4.$$

Here $\tau(n)^2 \leq d_4(n)$ prime-power by prime-power, and the fourfold divisor convolution has harmonic sum at most H_N^4.

For a lower bound on $\sum |v_k|^2$, put $F(n) = \prod_{p|n}(1-p^{-\alpha})^2$. Then

$$|v_n|^2 \geq n^{-1+2a} F(n).$$

The nonnegative squarefree-supported multiplicative function r with $r(p) = (1-p^{-\alpha})^{-2} - 1$ satisfies $1/F = r * 1$ and

$$\sum_{n \leq N} \frac{1}{F(n)} \leq N \prod_p \left(1 + \frac{r(p)}{p}\right) \leq N\mathfrak{C}_\alpha.$$

Indeed $r(p) \leq L_\alpha p^{-\alpha}$, and $\sum_p p^{-1-\alpha} \leq \zeta(1+\alpha) - 1 \leq 1/\alpha$. Cauchy–Schwarz on $N/2 < n \leq N$, which contains at least $N/2$ integers and omits coordinate 1, now gives

$$\sum_{n=2}^{N} |v_n|^2 \geq N^{-1+2a} \sum_{N/2 < n \leq N} F(n) \geq \frac{N^{2a}}{4\mathfrak{C}_\alpha}.$$

This proves the second term of (72) for the projection penalty in (70).

For the omitted evaluation norm set $\varepsilon = 1/\log N \le 2a$. Rankin's elementary comparison and the divisor Dirichlet series give

$$\sum_{n>N} |u_n|^2 \le \zeta(2) N^{-2a+\varepsilon} \zeta(1+\varepsilon)^4 \le e\zeta(2) N^{-2a} (1+\log N)^4.$$

Adding this tail and the projection penalty proves the bound. Finally the $n = 2$ term of (71), with $J_2(2) = 3$, is at least $(2^\sigma - 1)^2/3$, proving the last assertion. □

The extra null has no penalty in this limiting variance optimum. It still has its exact finite cost in (70). This theorem optimizes the leading tail variance, rather than the complete Poisson norm or the full gain: the minimizing coefficients may have large head norm, coefficient mass and remainder constants. Combining those quantities is the further arithmetic optimization required for a uniformly successful certificate.

G Corrected finite gain certificates

For coefficient selection, the original common arithmetic head must be kept with the tail. Fix $N \ge 4$ and write

$$\mathcal{H}_N = \{c \in \mathbb{C}^N : A_c(1) = A_c(q) = 0\}, \quad b_M c = m_M, \quad c^* G_M c = Q_M.$$

The entries of G_M are computed from the original states $B_n(e_d)$; these individual coordinate states are used only in finite sums. Complete norms below are asserted on $\mathcal{H}_N$.

Proposition G.1 (Paid head and norm envelope)**.** *Let V_N have entries $\gcd(d,e)^2/(12de)$ and put*

$$D_N = \sum_{d \le N} d^2, \quad \gamma = \frac{|q|}{12}\left(1 + \frac{|q+1|}{\sigma+1}\right),$$

$$r_{N,M} = \frac{2aD_N + a(2a+1)N/8}{M^2} + \frac{\gamma\sqrt{ND_N}}{(M+1)^2} + \frac{\gamma^2 D_N}{(M+1)^3},$$

$$T_+ = \left(\frac{2a}{M} + \frac{2a}{M^2}\right) V_N + r_{N,M} I, \qquad R_M = G_M + T_+.$$

These matrices are positive definite where needed, and for $c \in \mathcal{H}_N$,

$$Q_{\mathrm{tail}}(c) \le c^* T_+ c, \qquad Q(c) \le c^* R_M c.$$

In particular the matrix includes the complete norm, not only its head.

Proof. In (64) use $S_0 \le \sqrt{N}\,\|c\|$, $S_1 \le \sqrt{D_N}\,\|c\|$, $H = S_0/2$, $E = \gamma S_1$. Retain the $2aV/M^2$ term as a multiple of V_N. The other terms are bounded by $r_{N,M}\,\|c\|^2$ exactly as displayed. The gcd matrix V_N is positive definite by its invertible divisor factorization; also $r_{N,M} > 0$. Adding $G_M \succeq 0$ proves the claims. □

G.1 Choice of a complete norm envelope

The following corrected-gain proposition uses only a positive definite upper for the complete norm on the exact coefficient subspace. Besides the gcd envelope above, Theorem 3.1 supplies

$$R_M^{\mathrm{mix}} = G_M + U_R + N\delta_R I, \qquad Q(c) \le c^* R_M^{\mathrm{mix}} c \quad (c \in \mathcal{H}_N). \tag{73}$$

Here U_R is the positive matrix of (10); a positive extension using its divisor formula is equally valid on the anchored subspace. Every convex combination of complete upper matrices is again an upper. One may select the best certified scalar upper for a fixed trial, while retaining each matrix's actual congruence in a matrix solve.

In exact divisor coordinates $c = Ct$, $C_{dk} = d\mu(k/d)\mathbf{1}_{d|k}$, the full mixed upper on $t_1 = 0$ is

$$C^* G_M C + \frac{2a}{K}\,\mathrm{diag}_{k\ge 2}(W_{R,k}) + N\delta_R C^* C.$$

The compensated mean row transforms by the same C; its Euclidean error is charged against $\|Ct\|$. More generally, on any exact double-null basis $c = Jx$, the raw error $N\delta_R I$ becomes $N\delta_R J^* J$. Thus diagonalizing the tail does not replace the common head or the coefficient error by an unrelated identity metric.

For the rest of this section, R_M may denote any such proved positive definite complete envelope, including (73). The displayed gcd choice is its fixed-parameter specialization.

Proposition G.2 (Corrected gain and constrained scalar test)**.** *For any fixed finite $R \ge 1$ define the computable row and scalar*

$$b = b_M + t_{R,K}, \qquad \delta = C_R \left(\sum_{d\le N} d^{2R}\right)^{1/2} K^{-\alpha-R}\mathcal{A}_R, \qquad D = \delta^2 I.$$

For every nonzero $c \in \mathcal{H}_N$,

$$\frac{\left(\max\{0, |bc| - \delta\,\|c\|\}\right)^2}{c^* R_M c} \le g(A_c). \tag{74}$$

Fix a desired gain $\eta > 0$ and any injective basis matrix J for $\mathcal{H}_N$. A trial makes the left side of (74) *strictly exceed η if and only if*

$$\sup_{0<\theta<1} F(\theta) > 1, \qquad F(\theta) = bJ\left[J^*\left(\frac{D}{\theta} + \frac{\eta R_M}{1-\theta}\right)J\right]^{-1} J^* b^*. \tag{75}$$

If $bJ \ne 0$, F is strictly concave and has a unique maximizer, the unique zero of its strictly decreasing derivative. If $bJ = 0$, then $F \equiv 0$ and no trial passes this certificate.

Proof. Cauchy–Schwarz gives $S_R \le (\sum d^{2R})^{1/2}\,\|c\|$, so Theorem 7.1 gives $|m - bc| \le \delta\,\|c\|$. Proposition G.1 proves (74). The target inequality is $|bc| > \sqrt{c^* D c} + \sqrt{\eta c^* R_M c}$. For positive x, y,

$$(\sqrt{x} + \sqrt{y})^2 = \min_{0<\theta<1}\left(\frac{x}{\theta} + \frac{y}{1-\theta}\right).$$

For each θ, the constrained Rayleigh quotient of bc is the displayed inverse expression. The two existential quantifiers over c and θ commute, proving (75).

For completeness whiten $D_H = J^* D J$ and set $w = bJD_H^{-1/2}$, $S = \eta D_H^{-1/2} J^* R_M J D_H^{-1/2} > 0$. Then

$$F(\theta) = \theta(1-\theta) w[(1-\theta)I + \theta S]^{-1} w^*.$$

Spectral decomposition reduces its second derivative to positive weights times $-2s/[1+(s-1)\theta]^3$, $s>0$. Thus $F''<0$ if $w\neq 0$, while $F'(0)=ww^*>0$ and $F'(1)=-wS^{-1}w^*<0$. This proves uniqueness and permits certified derivative bisection; no secondary extrema need be excluded numerically. □

The worst-response norm and scalar optimization above are standard robust quadratic optimization, as in [19, 20]. The new input is the actual corrected arithmetic row and its complete error. Equivalence in (75) is to the paid certificate, not to the true finite-M gain. No numerical inverse is certified merely by writing its exact formula.

An alternative selection retains the center normalization explicitly. Let C have the three rows $A(1), A(q), A(\lambda)$, and $e=(0,0,1)^{\mathsf{t}}$. The columns $1,2,4$ form a Vandermonde matrix with nodes $2^{-1}, 2^{-q}, 2^{-\lambda}$ of distinct moduli, so C has full row rank. Standard constrained least squares gives

$$\min_{Cc=e} c^* R_M c = e^*[CR_M^{-1}C^*]^{-1}e.$$

This optimizes a complete upper model and therefore supplies an upper bound for the actual norm of the resulting polynomial. It does not replace the retained full head by its variance capacity.

For perspective, a mean-tail estimate using only the moment information $Q_{\mathrm{tail}}\le u=c^*T_+c$ and tail mass π_M gives

$$L_{\mathrm{mom}}(c)=\frac{\left(\max\{0,|b_Mc|-\sqrt{\pi_M u}\}\right)^2}{c^*G_Mc+u}.$$

This is sharp for abstract tails with those moment constraints: a constant tail state of opposing phase attains the adverse mean, or reduces it to zero when allowed. Such a state need not be realizable by arithmetic coefficients. Corollary 7.2 proves that its extremal alignment is asymptotically impossible for each fixed nonzero trial. This is the cancellation the corrected row preserves.

Reproducible bounded examples. The accompanying `verify_compensated_means.py` uses exact rational identities and 256-bit Arb/Acb enclosures. It checks 256 discrete pair covariances, seven complex whole-period energy identities, and orders $R=1,2,3$ for five actual arithmetic cases, using $A(s)=(1-2^{1-s})(1-2^{q-s})$. For these chosen coefficients one may use the sharper scalar error $C_RS_RK^{-\alpha-R}\mathcal{A}_R$ and the scalar norm upper from (64). The same complete denominator is used in the old and corrected certificates. Rounded displays of their order-two values are:

a	T	M	moment certificate	corrected certificate
1/8	0	512	.03133023	.04032653
1/4	3	512	.19172325	.20066055
2/5	0	512	.12157883	.12442350
2/5	3	2048	.28624639	.28717439
2/5	1000	8192	.08507373	.09517727

For the first case the paid mean radius falls from about .01185507 to $2.05425\cdot 10^{-8}$; at $T=1000$ it falls from .03760239 to .0002024034. All fifteen finite corrected cases strictly improve their moment comparison, and eight wrong-sign controls are rejected. The verification uses $\zeta(\lambda)A(\lambda)$ only as an independent truth check; the actual corrected certificate requires no zeta value. The explicit rising-factorial and coefficient-mass costs remain essential when parameters or support grow.

H A paired-wave scalar alternative

Here a short classical nonstationary argument supplies a scalar alternative to the mixed-order envelope. Under both nulls, the exact identity from Lemma 2.1 before integration by parts is

$$B_n = n^{-q} H_{0,c}(n) - q \int_n^\infty H_{0,c}(x) x^{-q-1} \, \mathrm{d}x.$$

Assume $T \geq 0$, conjugating q and all coefficients otherwise. For $\omega = 2\pi k/d$, $k \geq 1$, the two integrals

$$I_\pm = \int_n^\infty x^{-\sigma-1} e^{i(\pm\omega x - T \log x)} \, \mathrm{d}x$$

satisfy, when $n \geq N|q|/\pi$,

$$|I_\pm| \leq \frac{2n^{-\sigma-1}}{\omega \mp T/n}.$$

The quotients $x^{-\sigma}/(\omega x \mp T)$ are positive, decreasing to zero; integration by parts pays their endpoint and total variation. This is the standard monotone first-derivative estimate [6, 5].

Use the Abel Fourier series for the sawtooth only under the integral. Pair the signs before summing their coefficient magnitudes $1/(2\pi k)$:

$$\frac{|I_+| + |I_-|}{2\pi k} \leq \frac{d n^{-\sigma-1}}{\pi^2 (k^2 - \rho^2)}, \qquad \rho = dT/(2\pi n) \leq 1/2.$$

The majorant is summable and $\sum_{k \geq 1} (k^2 - 1/4)^{-1} = 2$ by telescoping. Abel averages have magnitude at most $1/2$; dominated convergence with $x^{-\sigma-1}$ returns the actual integral. The original inclusive endpoint $H_{0,c}(n)$ remains unchanged. Therefore

$$|B_n| \leq C_* S_0 n^{-\sigma}, \qquad C_* = \tfrac{1}{2} + 2/\pi < 5/4, \quad n \geq N|q|/\pi.$$

Summing the complete positive p_n weights gives

$$Q - Q_M \leq C_*^2 S_0^2 / K, \qquad |m - m_M| \leq C_* S_0 K^{-\alpha} \quad (K \geq N|q|/\pi).$$

This alternative credits classical height-uniform truncation; the mixed-order envelope retains the finer periodic covariance and resummed height-scale capacity.

Statements and declarations

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