Arithmetic cutoff bounds and sharp tail capacity for local Nyman–Beurling certificates

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Luminary Oddities

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Abstract

We determine the sharp tail capacity of anchored arithmetic trials in the whole-line Poisson norm at $\lambda = \alpha + iT$, $1/2 < \alpha < 1$. After a prescribed exterior null is imposed, the complete norm has an M^{-1} tail with a positive gcd-variance coefficient and an explicit period-free remainder. The normalized coefficient has exact infimum $[12(\zeta(2)\zeta(2\alpha)/|\zeta(1+\lambda)|^2 - 1)]^{-1}$. Finite constrained capacities approach this limit at an explicit rate uniform in T. A dyadic factor imposes the null while preserving arbitrarily much projection gain at a paid support cost, yielding complete finite arithmetic lower certificates. Complementarily, an exact weighted divisor dual gives $g_N(\lambda) \le K_\alpha N^{2-2\alpha}|\zeta(\lambda)|^2$ for all anchored complex coefficients supported on the integers at most N. The power is exact for the unrestricted head reconstruction norm. Montgomery's unconditional small-value theorem then gives a necessary support-cutoff growth rate for an all-height positive gain.

1 The local arithmetic problem

The Nyman–Beurling approximation problem admits local versions in which a projection gain determines a zero-free disk. Nikolski's approach and the extensions of Delaunay, Fricain, Mosaki and Robert [4] give the Hardy-space mechanism. We ask two complementary quantitative questions: how large an arithmetic cutoff must a uniform positive gain permit, and how can a finite arithmetic computation certify a gain in the complete norm?

Finite divisor inversion and arithmetic biorthogonality are classical. Balazard [1], building on Vasyunin's system, analyzes the harmonic dual and its quotient increments. The harmonic grouped identity in [2, Theorem 3] is the critical algebraic endpoint of the weighted identity below; related weighted native duals appear in [3]. Our recipient instead matches a complex interior evaluation to a Poisson weight, retaining uniformity in the translation T. The power-law evaluation bound and its support consequence are paired with a complete arithmetic lower-certificate construction. The latter's sharp tail and exact coefficient capacity follow from the classical gcd covariance of fractional parts and a diagonal Jordan transform; the covariance itself is credited to Franel and Landau, as recalled in [6]. The gcd quadratic form and its divisor factorization are classical; see, for example, Lindqvist and Seip [7]. The optimization here retains the particular evaluation and exterior-null rows of the local arithmetic problem. The exact capacity is Theorem 7.1; Corollary 7.2 gives its uniform finite-cutoff cost. The sharp tail in Theorem 6.1 connects that coefficient problem to complete arithmetic certificates, while Corollary 3.3 supplies the complementary support obstruction. The packet-kernel asymptotics of [8] concern a different, native evaluation problem; the local recipient used here is made explicit in that work's mathematical supplement.

Fix throughout

$$0 < a < \tfrac{1}{2}, \quad \alpha = \tfrac{1}{2} + a, \quad \sigma = \tfrac{1}{2} - a, \quad \lambda = \alpha + iT, \quad q = \sigma + iT, \quad T \in \mathbb{R}. \tag{1}$$

Let $\mathcal{A}_N$ be the Dirichlet polynomials

$$A(s) = \sum_{d \le N} c_d d^{-s}, \qquad A(1) = \sum_{d \le N} \frac{c_d}{d} = 0.$$

The coefficients may be complex and may depend on T. The integer N is a support cutoff, not the number of nonzero terms. Put

$$d_n = \sum_{d|n} c_d, \qquad B_n = \sum_{j \le n} d_j j^{-q}, \qquad p_n = n^{-2a} - (n+1)^{-2a}, \tag{2}$$

where $c_d = 0$ beyond the support. Define

$$m(A) = \sum_{n \ge 1} p_n B_n, \qquad Q(A) = \sum_{n \ge 1} p_n |B_n|^2, \qquad g(A) = \frac{|m(A)|^2}{Q(A)} \quad (A \neq 0),$$

and $g_N(\lambda) = \sup_{A \in \mathcal{A}_N \setminus \{0\}} g(A)$. For a trivial coefficient space the supremum is defined to be zero.

Proposition 1.1 (Complete recipient)**.** *For every anchored finite polynomial, all these moments exist and*

$$m(A) = \zeta(\lambda) A(\lambda), \tag{3}$$

$$Q(A) = \frac{a}{\pi} \int_{\mathbb{R}} \frac{|\zeta(1/2+it) A(1/2+it)|^2}{a^2 + (t-T)^2} \, \mathrm{d}t. \tag{4}$$

For $A \neq 0$, $Q(A) > 0$ and $0 \le g(A) \le 1$.

Proof. The summatory divisor sequence satisfies

$$D(x) = \sum_{j \le x} d_j = \sum_d c_d \lfloor x/d \rfloor = -\sum_d c_d \{x/d\}, \qquad |D(x)| \le S_0 := \sum_d |c_d|.$$

Partial summation shows that $\sum d_n n^{-s}$ converges on $\Re s > 0$, agrees there by continuation with $\zeta(s)A(s)$, and that B_n is bounded at fixed q. Thus both moments converge. Summation by parts against p_n gives

$$\sum_{n \le M} p_n B_n = \sum_{j \le M} d_j j^{-(q+2a)} - (M+1)^{-2a} B_M.$$

The last term tends to zero, proving (3) since $q + 2a = \lambda$.

Initially on $\Re s > 1$, the Mellin identity is

$$\int_1^\infty B_{\lfloor x \rfloor} x^{q-s} \frac{\mathrm{d}x}{x} = \frac{\zeta(s) A(s)}{s - q}.$$

Its bounded summatory representation continues to $\Re s > 1/2$. Mellin Plancherel on the boundary gives

$$\frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{\zeta(1/2+it) A(1/2+it)}{1/2 + it - q} \right|^2 \mathrm{d}t = \int_1^\infty |B_{\lfloor x \rfloor}|^2 x^{-2a} \frac{\mathrm{d}x}{x} = \frac{Q(A)}{2a},$$

which is (4). If $Q = 0$, positivity of p_n forces every B_n, hence every d_n and c_n, to vanish. Finally $\sum p_n = 1$, so Cauchy–Schwarz proves $|m|^2 \le Q$. □

Minimizing $\sum p_n |1 - z B_n|^2$ over $z \in \mathbb{C}$ gives $1 - g(A)$. This identifies g_N as an optimized projection gain, without an independence assumption on a chosen trial dictionary.

2 Exact evaluation duals

Write μ for the Möbius function and

$$M_x(z) = \sum_{m \le x} \mu(m) m^{-z}.$$

For $1 \le n \le N$ set

$$v_n = n^{-2a} M_{N/n}(\lambda), \qquad v_{N+1} = 0, \quad \ell_n = v_n - v_{n+1}, \quad \mathcal{D}_N = \sum_{n \le N} \frac{|\ell_n|^2}{p_n}. \tag{5}$$

Theorem 2.1 (Evaluation-first dual)**.** *For every coefficient vector supported at most N,*

$$A(\lambda) = \sum_{n \le N} \ell_n B_n.$$

The squared norm of this functional on the unrestricted first-head space with norm $\sum_{n \le N} p_n |B_n|^2$ is exactly $\mathcal{D}_N$. For the anchored complete recipient,

$$g_N(\lambda) \le \min\{1, |\zeta(\lambda)|^2 \mathcal{D}_N\}. \tag{6}$$

Proof. Finite inversion gives

$$c_k = \sum_{d \mid k} \mu(k/d) d^q (B_d - B_{d-1}), \qquad B_0 = 0.$$

Substitution in $\sum c_k k^{-\lambda}$ yields $\sum_{d \le N} d^{q-\lambda} M_{N/d}(\lambda)(B_d - B_{d-1})$. Since $q - \lambda = -2a$, summation by parts gives the stated row. Weighted Cauchy–Schwarz proves the norm bound, with equality in the unrestricted head at $B_n = \overline{\ell_n}/p_n$. The full Q is at least its positive first head, so (3) proves (6). □

Anchoring can be retained exactly, rather than discarded in this upper bound. Put

$$w_n = n^{q-1} M_{N/n}(1), \quad w_{N+1} = 0, \quad h_n = w_n - w_{n+1}.$$

The same inversion gives $A(1) = \sum h_n B_n$. In particular $h \neq 0$, since $h_N = N^{q-1}$. The exact squared head norm restricted to anchoring is

$$\mathcal{D}_N^{\mathrm{anch}} = \mathcal{D}_N - \frac{\left|\sum_{n \le N} \ell_n \overline{h_n}/p_n\right|^2}{\sum_{n \le N} |h_n|^2/p_n}. \tag{7}$$

This follows by orthogonal projection after setting $x_n = \sqrt{p_n} B_n$. Thus $\mathcal{D}_N$ in (6) may be replaced by $\mathcal{D}_N^{\mathrm{anch}}$. Neither head norm is asserted equal to the optimized gain in the complete arithmetic recipient.

3 A log-free power law and a necessary cutoff cost

Let $\delta = 1 - \alpha = \sigma$ and $b = 1 - 2a = 2\delta$. For $K \ge 1$ define a finite sequence $H_K(n) = n^{-2a} \mathbf{1}_{n \le K}$ and its forward difference $\Delta H_K(n) = H_K(n) - H_K(n+1)$.

Lemma 3.1 (Rank-one plus diagonal Gram)**.** *In the difference norm with weights* $1/p_n$*,*

$$\langle \Delta H_K, \Delta H_L \rangle_{1/p} = 1 \quad (K \neq L), \qquad d_K := \|\Delta H_K\|_{1/p}^2 = 1 - K^{-2a} + \frac{K^{-4a}}{p_K}.$$

Writing

$$u_K = \sum_{\lfloor N/m \rfloor = K} \mu(m) m^{-\lambda},$$

one consequently has the exact identity

$$\mathcal{D}_N = |M_N(\lambda)|^2 + \sum_{K \leq N} (d_K - 1)|u_K|^2. \tag{8}$$

Proof. The difference ΔH_K equals p_n for $n < K$, equals K^{-2a} at $n = K$, and vanishes after that. For $K < L$ the cross product is $\sum_{n<K} p_n + K^{-2a} = 1$. The diagonal formula follows similarly. Now $v = \sum_{m \leq N} \mu(m) m^{-\lambda} H_{\lfloor N/m \rfloor}$. Expand its difference norm and group equal floor values. The all-ones part is $|\sum u_K|^2 = |M_N(\lambda)|^2$. □

Theorem 3.2 (Uniform power law)**.** *For every* $T \in \mathbb{R}$ *and integer* $N \geq 1$*,*

$$N^b \left(\frac{1}{2a} - \frac{1}{N} \right) \leq \mathcal{D}_N \leq K_a N^b, \tag{9}$$

where one admissible explicit constant is

$$K_a = \delta^{-2} + \frac{2^{2a}}{a}(2^{2\alpha+2} + 1)\zeta(2).$$

In particular $g_N(\lambda) \leq K_a N^{2-2\alpha} |\zeta(\lambda)|^2$*. The exponent in* (9) *is sharp for* $\mathcal{D}_N$*.*

Proof. The integral formula for p_K gives

$$a 2^{-2a} K^{-2a-1} \leq p_K \leq 2a K^{-2a-1}, \qquad 0 < d_K - 1 \leq \frac{2^{2a}}{a} K^b.$$

Also $|M_N(\lambda)| \leq \sum_{m \leq N} m^{-\alpha} \leq N^\delta/\delta$. If $K \leq \sqrt{N}$, the interval $N/(K+1) < m \leq N/K$ contains at most $2N/K^2$ integers. Hence

$$|u_K| \leq 2^{\alpha+1} N^\delta K^{\alpha-2}, \qquad K^b |u_K|^2 \leq 2^{2\alpha+2} N^b / K^2.$$

If $K > \sqrt{N}$, that interval has length less than one. Each nonempty group contains just one m, and its contribution is at most N^b/m^2. Summing both regimes proves the upper bound via (8). This is the weighted quotient-increment split underlying the related harmonic estimates of [1].

All terms in (8) are nonnegative. The group $K = N$ contains only $m = 1$, so $u_N = 1$ and

$$\mathcal{D}_N \geq d_N - 1 = \frac{N^{-4a}}{p_N} - N^{-2a} \geq N^b \left(\frac{1}{2a} - \frac{1}{N} \right).$$

This sharpness statement concerns the unrestricted head functional; anchoring and the full tail remain present in the gain inequality. □

Corollary 3.3 (All-height gain requires growing support)**.** *Fix $\alpha \in (1/2, 1)$ and $\eta > 0$. If an arbitrary integer schedule $N(t) \geq 2$ satisfies $g_{N(t)}(\alpha + it) \geq \eta$ for all sufficiently large t, then along an unbounded sequence t_j,*

$$\log N(t_j) \geq \frac{\sqrt{\alpha - 1/2}}{20(1-\alpha)} \frac{(\log t_j)^{1-\alpha}}{(\log\log t_j)^{\alpha}} + O_{\alpha,\eta}(1). \tag{10}$$

Thus every cutoff with logarithm of smaller order than the displayed height scale fails a fixed positive gain requirement infinitely often.

Proof. Montgomery's unconditional Theorem 1 [5], at angle π, gives for every sufficiently large X a height $X^{(\alpha-1/2)/3} \leq t \leq X$ such that

$$|\zeta(\alpha + it)| \leq \exp\left[-\frac{\sqrt{\alpha - 1/2}}{20} \frac{(\log X)^{1-\alpha}}{(\log\log X)^{\alpha}}\right].$$

The scale is eventually increasing, so it can be replaced by its value at t. These heights are unbounded. Apply Theorem 3.2 and take logarithms of $\eta \leq K_a N(t)^b |\zeta(\alpha+it)|^2$. Since $b = 2(1-\alpha)$, this is (10). □

The source's sufficiently-large onset is not replaced by a numerical height here. The statement fixes α, permits all complex coefficient choices at every height, and concerns the largest permitted integer index rather than coefficient cardinality.

4 Null adaptation without loss of the closed trial space

The support lower cost does not construct a successful trial. We next give a complete way to certify any proposed growing trial.

Theorem 4.1 (Costed exterior-null adaptation)**.** *For nonzero $A \in \mathcal{A}_N$ and integer $k \geq 1$, set*

$$\varepsilon = 2^{-ka}, \qquad A_k(s) = A(s)(1 - 2^{k(q-s)}).$$

Then $A_k(1) = A_k(q) = 0$ and its support is at most $2^k N$. Moreover

$$Q(A_k - A) = \varepsilon^2 Q(A), \tag{11}$$

$$(1-\varepsilon)^2 Q(A) \leq Q(A_k) \leq (1+\varepsilon)^2 Q(A), \tag{12}$$

$$m(A_k) = (1-\varepsilon^2) m(A), \tag{13}$$

$$(1-\varepsilon)^2 g(A) \leq g(A_k) \leq (1+\varepsilon)^2 g(A). \tag{14}$$

The arithmetic images satisfying both nulls have the same closed span as all anchored arithmetic images, at every fixed a, T.

Proof. On $\Re s = 1/2$, the multiplier $2^{k(q-s)}$ has constant modulus ε. Equation (4) proves (11); the triangle inequality proves the two norm bounds. At λ its value is $2^{-2ka} = \varepsilon^2$, proving the mean identity and the gain bounds. The two nulls and support claim are immediate, with any coincident integer coefficients added. As $k \to \infty$, the full-norm error tends to zero, proving equality of closures. The reverse inclusion holds because the doubly null polynomials are already anchored. □

The transformed coefficient mass obeys

$$S_0(A_k) \leq (1 + 2^{k\sigma}) S_0(A). \tag{15}$$

Thus small gain loss is accompanied by an explicit cost both in support and in coefficient mass; the construction is not cost-free as $a \to 0$.

5 Complete finite arithmetic lower certificates

Theorem 5.1 (Tail-paid lower gain)**.** *Suppose $A \neq 0$ and $A(1) = A(q) = 0$, and put*

$$K = \left(1 + \frac{|q|}{\sigma}\right) S_0(A), \qquad m_M = \sum_{n \le M} p_n B_n, \quad Q_M = \sum_{n \le M} p_n |B_n|^2.$$

Then for every integer $M \ge 1$,

$$0 \le Q - Q_M \le \frac{K^2}{M+1}, \tag{16}$$

$$|m - m_M| \le \frac{K}{(M+1)^\alpha}. \tag{17}$$

Consequently the finite arithmetic quantity

$$L_M(A) = \frac{\left(\max\{0, |m_M| - K/(M+1)^\alpha\}\right)^2}{Q_M + K^2/(M+1)} \tag{18}$$

satisfies $0 \le L_M(A) \le g(A)$ and tends to $g(A)$ as $M \to \infty$. No zeta value is needed to evaluate (18)*.*

Proof. Partial summation and boundedness of D give

$$\lim_{n\to\infty} B_n = q \int_1^\infty D(x) x^{-q-1}\,\mathrm{d}x = \zeta(q) A(q) = 0.$$

Subtracting this limit yields

$$B_n = D(n) n^{-q} - q \int_n^\infty D(x) x^{-q-1}\,\mathrm{d}x, \qquad |B_n| \le K n^{-\sigma}.$$

Since $\sum_{n>M} p_n = (M+1)^{-2a}$, positivity and $2a + 2\sigma = 1$, $2a + \sigma = \alpha$ prove both tail bounds. The triangle inequality for m and upper bound for Q prove the certificate; convergence follows from the vanishing tails and $Q > 0$. □

Combining Theorems 4.1 and 5.1 gives a complete finite hierarchy at a fixed center. To approach the limiting optimized gain, first choose a finite anchored A near its supremum, then k to retain its gain, then M to approximate $g(A_k)$. For a monotone sequence of lower bounds, retain the largest certificate obtained so far. Completeness is an exhaustion statement; a successful uniform schedule must additionally bound its support, mass and tail parameters over the claimed height range.

As a literal example,

$$A(s) = (1 - 2^{1-s})(1 - 2^{q-s})$$

uses only $1, 2, 4$ and satisfies both nulls. Its center value is $(1 - 2^{1-\lambda})(1 - 2^{-2a}) \neq 0$ because $|2^{1-\lambda}| = 2^{1-\alpha} > 1$. It demonstrates the finite construction; its fixed cutoff remains subject to Corollary 3.3.

6 The sharp complete tail and a period-free remainder

The elementary tail upper in (16) can be refined to its leading coefficient. For a finite anchored coefficient vector put

$$V(c) = \frac{1}{12} \sum_{d,e} c_d \overline{c_e} \frac{\gcd(d,e)^2}{de}, \qquad S_1 = \sum_d d|c_d|, \qquad H = \frac{S_0}{2}. \tag{19}$$

Let $J_2(k) = k^2 \prod_{p|k}(1 - p^{-2})$ be the second Jordan function.

Theorem 6.1 (Sharp null-adapted tail)**.** *For every nonzero finite A with $A(1) = A(q) = 0$, $V(c) > 0$. Set*

$$E = \frac{|q|S_1}{12}\left(1 + \frac{|q+1|}{\sigma+1}\right), \qquad C_0 = 2a(V(c) + S_1^2) + \frac{a(2a+1)H^2}{2}.$$

Then, for every integer $M \geq 1$,

$$\left|\sum_{n>M} p_n |B_n|^2 - \frac{2aV(c)}{M}\right| \leq \frac{C_0}{M^2} + \frac{2HE}{(M+1)^2} + \frac{E^2}{(M+1)^3}. \tag{20}$$

In particular the M^{-1} order is exact for every fixed nonzero trial.

Proof. The continuous period mean of $D(x)$ is $\overline{D} = -\frac{1}{2}\sum c_d$. Define

$$h(x) = D(x) - \overline{D} = \sum_d c_d(1/2 - \{x/d\}), \qquad P(x) = -\frac{1}{2}\sum_d c_d d B_2(\{x/d\}),$$

where $B_2(u) = u^2 - u + 1/6$. The periodic continuous primitive P satisfies $P' = h$ almost everywhere, $|h| \leq H$ and $|P| \leq S_1/12$. The zero limit of B_n and integration by parts give

$$B_n = h(n)n^{-q} + qP(n)n^{-q-1} - q(q+1)\int_n^\infty P(x)x^{-q-2}\,\mathrm{d}x.$$

Thus $B_n = h(n)n^{-q} + R_n$ with

$$|R_n| \leq En^{-\sigma-1}. \tag{21}$$

We require the discrete rather than continuous fractional-part covariance. Write $f_d(n) = 1/2 - \{n/d\}$ and $r = \gcd(d,e)$. Conditional on $n \bmod r = z$, the residues modulo d and e are independent on their admissible classes by the Chinese remainder theorem. Their conditional means are $(r/d)f_r(z)$ and $(r/e)f_r(z)$. Since $r^{-1}\sum_{z=0}^{r-1} f_r(z)^2 = 1/12 + 1/(6r^2)$, the exact discrete period mean is

$$\operatorname{av}_n f_d(n)f_e(n) = \frac{\gcd(d,e)^2}{12de} + \frac{1}{6de}. \tag{22}$$

The extra term cancels in the coefficient quadratic form by anchoring. Hence $\operatorname{av}|h(n)|^2 = V(c)$. The familiar continuous gcd covariance [6] is thus used here with its necessary discrete correction. The divisor identity for J_2 gives

$$12V(c) = \sum_k J_2(k)\left|\sum_{k|d}\frac{c_d}{d}\right|^2. \tag{23}$$

For a largest index with $c_d \neq 0$, the corresponding term is positive, which proves strict positivity.

For each pair (d,e), the centered sequence in (22) has mean zero, period $\operatorname{lcm}(d,e)$, and terms of modulus at most $1/2$. Therefore, writing $z_n = |h(n)|^2 - V(c)$,

$$\left|\sum_{n\leq L} z_n\right| \leq \frac{1}{2}\sum_{d,e}|c_d c_e|\operatorname{lcm}(d,e) \leq \frac{S_1^2}{2} \qquad (L \geq 1).$$

Abel summation gives $|\sum_{n>M} z_n/n^2| \leq S_1^2/(M+1)^2$. Also

$$|p_n - 2an^{-2a-1}| \leq a(2a+1)n^{-2a-2}, \quad \frac{1}{M+1} \leq \sum_{n>M} n^{-2} \leq \frac{1}{M}.$$

It follows that

$$\left| \sum_{n>M} p_n |h(n)|^2 n^{-2\sigma} - \frac{2aV(c)}{M} \right| \le \frac{C_0}{M^2}.$$

Finally (21) bounds the difference of the squared terms by $2HEn^{-2\sigma-1} + E^2 n^{-2\sigma-2}$. Summing against the full p_n and using their telescoping tail proves (20). No common-period length appears in its constants. □

The denominator of (18) may use the smaller of the proved upper tails $K^2/(M+1)$ and $2aV/M$ plus the right-hand side of (20). Subtracting a known leading term can improve an enclosure, but additional zeros at q alone do not change the nonzero M^{-1} coefficient.

7 Exact all-support tail capacity

The coefficient in Theorem 6.1 itself has a sharp optimization independent of the full head norm. For complex z define $J_z(k) = \sum_{d|k} \mu(k/d) d^z$.

Theorem 7.1 (Sharp tail-variance capacity)**.** *At every fixed center* (1)*, put*

$$S_\infty = \frac{\zeta(2)\zeta(2\alpha)}{|\zeta(1+\lambda)|^2} - 1 > 0.$$

Then

$$\inf_{\substack{A \text{ finite},\ A(1)=A(q)=0 \\ A(\lambda)=1}} V(c) = \frac{1}{12 S_\infty}. \tag{24}$$

Every zeta factor displayed in S_∞ is in its absolutely convergent half-plane. The infimum is approached by explicit finite polynomials.

Proof. Put $x_d = c_d/d$ and $t_k = \sum_{k|d} x_d$. On support at most N the inverse is $x_d = \sum_{m \le N/d} \mu(m) t_{dm}$, and

$$A(s) = \sum_{k \le N} t_k J_{1-s}(k), \qquad A(1) = t_1, \qquad 12V(c) = \sum_{k \le N} J_2(k) |t_k|^2. \tag{25}$$

Thus anchoring is $t_1 = 0$. On coordinates $2 \le k \le N$ define

$$u_k = \frac{J_{1-\lambda}(k)}{\sqrt{J_2(k)}}, \qquad v_k = \frac{J_{1-q}(k)}{\sqrt{J_2(k)}}.$$

The exact finite capacity is

$$S_N = \sum_{k=2}^{N} |u_k|^2 - \frac{|\sum_{k=2}^{N} u_k \overline{v_k}|^2}{\sum_{k=2}^{N} |v_k|^2}. \tag{26}$$

The denominator is nonzero for $N \ge 2$ since $2^{1-q} \ne 1$. When $S_N > 0$, the exact minimum of V under both nulls and $A(\lambda) = 1$ is $1/(12S_N)$, by projecting the evaluation row off the null row in the coordinates $y_k = \sqrt{J_2(k)} t_k$. Writing $C_N = \sum u_k \overline{v_k}$, an explicit minimizer is

$$y_k = \frac{\overline{u_k} - \overline{v_k}\, \overline{C_N} / \sum |v_j|^2}{S_N}.$$

It satisfies $\sum v_k y_k = 0$, $\sum u_k y_k = 1$, and $\sum |y_k|^2 = 1/S_N$. Formula (25) then constructs the coefficients. If $S_N = 0$, these normalized constraints are infeasible rather than an occasion to invert a singular matrix.

For $z = 1 - \lambda = \sigma - iT$, the prime-power formulas give

$$1 + \sum_{j \geq 1} \frac{|J_z(p^j)|^2}{J_2(p^j)} = 1 + \frac{|p^z - 1|^2}{(p^2 - 1)(1 - p^{2\sigma - 2})}$$
$$= \frac{|1 - p^{z-2}|^2}{(1 - p^{-2})(1 - p^{2\sigma - 2})}.$$

All factors converge absolutely since $2 - 2\sigma = 2\alpha > 1$ and $2 - \sigma > 1$. Multiplicativity and the positive series imply

$$\sum_{k \geq 1} \frac{|J_{1-\lambda}(k)|^2}{J_2(k)} = \frac{\zeta(2)\zeta(2\alpha)}{|\zeta(1 + \lambda)|^2}. \tag{27}$$

In particular $u \in \ell^2$ with squared norm $S_\infty > 0$ after omitting coordinate 1. Cauchy–Schwarz in (25) already proves $V \geq |A(\lambda)|^2/(12 S_\infty)$ for every finite anchored polynomial, even without the q-null condition.

In contrast, $v \notin \ell^2$. At primes,

$$\frac{|J_{1-q}(p)|^2}{J_2(p)} \geq (1 - 2^{-\alpha})^2 p^{2\alpha - 2} \geq \frac{(1 - 2^{-\alpha})^2}{p},$$

and the reciprocal primes have divergent sum. Thus $\|v_{2:N}\| \to \infty$. The normalized rows converge weakly to zero: each fixed finite head divided by this norm tends to zero, and Cauchy–Schwarz bounds its pairing with the remaining part of u by that part's ℓ^2 norm. Consequently

$$\frac{|\sum_{k=2}^N u_k \overline{v_k}|}{\|v_{2:N}\|} \to 0, \qquad S_N \to S_\infty.$$

For all sufficiently large N, $S_N > 0$ and the finite minimizers therefore approach the lower bound, proving (24). □

Corollary 7.2 (Uniform finite-cutoff capacity)**.** *Let*

$$L_\alpha = \frac{2 - 2^{-\alpha}}{(1 - 2^{-\alpha})^2}, \qquad \mathfrak{C}_\alpha = \exp(L_\alpha/\alpha).$$

For every real T and every integer $N \geq 2$ with $\log N \geq 1/(2a)$,

$$0 \leq S_\infty - S_N \leq N^{-2a} \left[e\zeta(2)(1 + \log N)^4 + 4\mathfrak{C}_\alpha \zeta(2)^2 (1 + \log N)^8 \right]. \tag{28}$$

Moreover $S_\infty \geq (2^\sigma - 1)^2/3$ uniformly in T. Thus a cutoff making the right side smaller than this positive constant guarantees feasibility of the normalized finite constraints for every T.

Proof. Write $\tau(n)$ for the divisor function and $H_N = \sum_{n \leq N} 1/n$. The elementary estimates $|J_z(n)| \leq n^{\Re z} \tau(n)$ and $J_2(n) \geq n^2/\zeta(2)$ imply

$$\left| \sum_{k=2}^N u_k \overline{v_k} \right| \leq \zeta(2) \sum_{k \leq N} \frac{\tau(k)^2}{k} \leq \zeta(2) H_N^4.$$

Here $\tau(n)^2 \le d_4(n)$ prime-power by prime-power, and the fourfold divisor convolution has harmonic sum at most H_N^4.

For a lower bound on $\sum |v_k|^2$, put $F(n) = \prod_{p|n}(1-p^{-\alpha})^2$. Then

$$|v_n|^2 \ge n^{-1+2a}F(n).$$

The nonnegative squarefree-supported multiplicative function r with $r(p) = (1-p^{-\alpha})^{-2}-1$ satisfies $1/F = r * 1$ and

$$\sum_{n\le N}\frac{1}{F(n)} \le N\prod_p\left(1+\frac{r(p)}{p}\right) \le N\mathfrak{C}_\alpha.$$

Indeed $r(p) \le L_\alpha p^{-\alpha}$, and $\sum_p p^{-1-\alpha} \le \zeta(1+\alpha)-1 \le 1/\alpha$. Cauchy–Schwarz on $N/2 < n \le N$, which contains at least $N/2$ integers and omits coordinate 1, now gives

$$\sum_{n=2}^{N}|v_n|^2 \ge N^{-1+2a}\sum_{N/2<n\le N}F(n) \ge \frac{N^{2a}}{4\mathfrak{C}_\alpha}.$$

This proves the second term of (28) for the projection penalty in (26).

For the omitted evaluation norm set $\varepsilon = 1/\log N \le 2a$. Rankin's elementary comparison and the divisor Dirichlet series give

$$\sum_{n>N}|u_n|^2 \le \zeta(2)N^{-2a+\varepsilon}\zeta(1+\varepsilon)^4 \le e\zeta(2)N^{-2a}(1+\log N)^4.$$

Adding this tail and the projection penalty proves the bound. Finally the $n=2$ term of (27), with $J_2(2)=3$, is at least $(2^\sigma-1)^2/3$, proving the last assertion. □

The extra null has no penalty in this limiting variance optimum. It still has its exact finite cost in (26). This theorem optimizes the leading tail variance, rather than the complete Poisson norm or the full gain: the minimizing coefficients may have large head norm, coefficient mass and remainder constants. Combining those quantities is the further arithmetic optimization required for a uniformly successful certificate.

8 Projection disks and arithmetic searches

For completeness, let $\Omega = \{\Re s > 1/2\}$ and $k_\lambda(s) = (s+\overline{\lambda}-1)^{-1}$ be its Hardy kernel. The image of A is $f_A(s) = a\zeta(s)A(s)/(s-q)$, with $\|f_A\|_{H^2}^2 = aQ(A)/2$. The normalized squared projection of ak_λ onto the span of these images is g_N. If $\zeta(\rho)=0$, $\rho\in\Omega$, every image vanishes at ρ. Projecting the kernel off k_ρ therefore gives

$$g_N(\lambda) \le \left|\frac{\rho-\lambda}{\rho+\overline{\lambda}-1}\right|^2.$$

This is the classical local projection mechanism of [4]. A proved lower certificate $L>0$ thus excludes zeros from the disk

$$\left|\frac{s-\lambda}{s+\overline{\lambda}-1}\right| < \sqrt{L}.$$

An implementation must use enclosures for the finite sums in (18), and pay the coefficient growth in (15). The upper dual in (7) and the lower certificate measure opposite sides of the same complete arithmetic problem. The missing constructive input for a prescribed all-height disk width is a family whose certified lower gains exceed its geometric threshold throughout that range.

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