# All-depth root-Laguerre positivity with nonreal Fourier zeros

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**Abstract**

We give an explicit admissible Fourier kernel for which every continuous root-coordinate Laguerre iterate is positive and uniformly strictly log-concave, although its Fourier transform has nonreal zeros. The fixed kernel is $\phi(s) = \exp(-e^{s^2} - 999s^2)$. Its Fourier transform is entire of exact order one, and a whole-line rational Gaussian-moment enclosure proves that the normalized sixth cumulant is less than $-10^{-18}$. The real-zero canonical product would require that cumulant to be nonnegative. Exact quadratic-tilt invariance explains why the entire root hierarchy misses this obstruction. We also give a general Gaussian-localization formula for higher cumulants, including an effective finite-jet error bound. Newman's Gaussian-damping classification already implies existence of a bad tilt for the underlying double-exponential family; the present conclusion is an explicit certified all-depth counterexample. It concerns general admissible kernels.

## 1 The question and the explicit counterexample

For a positive smooth function of a real variable $u$, set

$$\mathcal{T}f = (f')^2 - ff'' = -f^2(\log f)''.$$

Given an even spatial kernel $\phi$, its root-coordinate profile is $S(u) = \phi(\sqrt{u})$ for $u \geq 0$. All derivatives in $\mathcal{T}$ act on $u$, not on the spatial variable $s$. For $f > 0$, positivity of $\mathcal{T}f$ is equivalent to the differential condition $(\log f)'' < 0$ at each point.

The question considered here is whether full spatial admissibility and positivity with logarithmic concavity at *every* continuous root-iteration depth force the Fourier transform to have only real zeros. We answer this implication negatively with one fixed kernel. Throughout the paper,

$$F(z) = \int_{\mathbb{R}} \phi(s) e^{izs}\, ds.$$

We call a spatial kernel admissible here if it is positive, even and $C^\infty$, strictly decreasing for $s > 0$, and for some $\varepsilon > 0$ and every fixed $j \geq 0$, $\phi^{(j)}(s) = O(e^{-|s|^{2+\varepsilon}})$ as $|s| \to \infty$. Our example has much stronger tails.

**Theorem 1.1.** *Let $\phi(s) = \exp(-e^{s^2} - 999s^2)$ and let $S_0(u) = \phi(\sqrt{u})$, $S_{r+1} = \mathcal{T}S_r$ on $[0, \infty)$. Then:*

1. *$\phi$ is admissible, extends to an entire function, and $(\log \phi)'' \leq -2000$ on $\mathbb{R}$. For every fixed $j$ and every $p > 0$, $\phi^{(j)}(s) = O(e^{-|s|^p})$.*
2. *For every integer $r \geq 0$,*

$$S_r(u) = 2^{2^r - r - 1} \exp\big[(-998 \cdot 2^r - 1)u - 2^r e^u\big] > 0, \qquad (\log S_r)'' = -2^r e^u \leq -2^r. \tag{1}$$

*These statements include $u = 0$ and the entire unbounded half-line.*

3. *$F$ is even, real on $\mathbb{R}$, entire of exact order one, and has a nonreal zero. It has no purely imaginary zeros, so its nonreal zeros include a quartet off both axes.*

The quantifiers in this theorem fix one kernel before choosing the depth. This differs from prescribing a finite depth and then choosing a kernel. The proof of nonreal zeros is the exact cumulant certificate in Section 3; no numerical complex-zero approximation is needed.

### 1.1 Predecessors and the contribution boundary

Newman [1, Theorem 2] classifies even finite positive measures for which *every* Gaussian damping $e^{-bs^2}$, $b > 0$, has a real-zero Fourier transform. His product densities satisfy the quartic lower-tail bound in equation (1.9), derived from (1.10). The density $e^{-Ae^{s^2}}$, $A > 0$, violates that lower bound. Thus his theorem already implies that some negative Gaussian tilt of this density has nonreal Fourier zeros. That existence principle and the canonical-product cumulant sign used below are classical inputs.

The author's finite-depth paper [4] already displays the exact all-depth root profile $e^{-Ae^u}$, without establishing nonreal Fourier zeros for it. The present contribution is the explicit fixed quadratic tilt, its reader-complete rational certificate, and the resulting simultaneous all-depth shape/nonreal-zero conclusion. The Gaussian-localization identity in Section 4 supplies a reusable diagnostic using classical completion of the square and approximation by Gaussian averages.

Csordas–Dimitrov [2, Lemma 2.3, Theorem 2.4] and Dimitrov–Lucas [3, Theorem 1] derive ordinary and double Turán or cubic-Jensen conclusions from precise root-shape hypotheses. Those statements concern particular normalized moment sequences. Applying a moment theorem to a new function $\mathcal{T}^r S$ does not identify its moment sequence with an iterated Turán transform of the original sequence. No such identification is used here. In particular, the present generic example does not disprove the theta-specific second-level conjecture in [2, Problem 3.3].

## 2 The exact hierarchy and spatial admissibility

**Lemma 2.1.** *For real $B$ and every smooth $f$,*

$$\mathcal{T}(e^{Bu}f) = e^{2Bu}\mathcal{T}f, \qquad \mathcal{T}^r(e^{Bu}f) = e^{2^rBu}\mathcal{T}^r f$$

*at every depth where the derivatives exist. In particular all logarithmic curvatures of positive iterates are unchanged by this tilt.*

*Proof.* Differentiating $e^{Bu}f$ twice cancels the $B^2f^2$ and $2Bff'$ terms in $(f')^2 - ff''$. Induction proves the second identity; the logarithm of its exponential multiplier has zero second derivative. □

For $A > 0$ and $B \in \mathbb{R}$, direct induction gives

$$\mathcal{T}^r\left(e^{Bu-Ae^u}\right) = A^{2^r-1}2^{2^r-r-1}e^{(2^rB+2^r-1)u-2^rAe^u}. \tag{2}$$

Indeed if $f = Ce^{bu-ae^u}$ then $\mathcal{T}f = C^2ae^{(2b+1)u-2ae^u}$. Thus $a_r = 2^rA$, $b_r = 2^rB + 2^r - 1$, and $C_r = A^{2^r-1}2^{2^r-r-1}$ solve all three recurrences, including $C_0 = 1$. This extends the displayed un-tilted profile in [4]; the all-depth recurrence itself is an inherited ingredient. Taking $A = 1$, $B = -999$ proves (1).

More generally $\phi_{A,B}(s) = e^{-Ae^{s^2}+Bs^2}$ is positive, even and entire. For $B < A$,

$$(\log\phi_{A,B})''(s) = 2B - 2Ae^{s^2}(1 + 2s^2) \le -2(A - B) < 0.$$

Its first derivative is strictly negative for $s > 0$. Every fixed derivative is $\phi_{A,B}$ times a polynomial in $s$ and $e^{s^2}$. The negative double exponential dominates these factors and every fixed power of $|s|$, proving all the stated real-axis derivative tails. The root formula is entire in $u$, so its endpoint requires no singular limiting differentiation.

**Lemma 2.2.** *For $A > 0$ and $B \in \mathbb{R}$, the Fourier transform of $\phi_{A,B}$ is entire of exact order one and infinite exponential type.*

*Proof.* Every real exponential moment is finite, so compact-set domination justifies differentiation of the Fourier integral at every complex $z$. For any $p > 1$, $\phi_{A,B}(s) \le C_p e^{-2|s|^p}$. Young's inequality gives $r|s| \le |s|^p + C_p' r^{p/(p-1)}$. Consequently

$$\max_{|z|\le r} |F(z)| \le C_p \left( \int_{\mathbb{R}} e^{-|s|^p}\, ds \right) e^{C_p' r^{p/(p-1)}}.$$

Its order is at most $p/(p-1)$ for every $p > 1$, hence at most one. For any fixed $a > 0$, positivity on $[-a-1, -a]$ gives $F(iy) \ge c_a e^{ay}$ for $y > 0$, with $c_a > 0$. This excludes order less than one; allowing arbitrary $a$ also excludes finite exponential type. Evenness and reality follow from the integral. Finally $F(iy) = \int \phi(s) e^{-ys}\, ds > 0$ for all real $y$. □

## 3 An exact nonreal-zero certificate

We prove the nonreal-zero assertion in Theorem 1.1 using a negative sixth cumulant. The moment bounds below cover the entire real integration domain. They require neither a numerical zero approximation nor a truncated quadrature.

Normalize the kernel $\phi(s) = \exp(-e^{s^2} - 999s^2)$ to a probability density and denote its even moments by $\mu_{2j}$. Put

$$C = 1000, \qquad \varepsilon = C^{-1}, \qquad x = \sqrt{C}\, s, \qquad R(t) = e^t - 1 - t.$$

The constant $e^{-1}$ in

$$\phi(s) = e^{-1} e^{-Cs^2} e^{-R(s^2)}$$

cancels on normalization. Define the unnormalized scaled moments

$$I_j = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} x^{2j} e^{-x^2} e^{-R(\varepsilon x^2)}\, dx, \qquad a_j = \frac{(2j)!}{4^j j!}. \tag{3}$$

Thus $a_j$ is the corresponding Gaussian moment and

$$\mu_{2j} = \frac{I_j}{C^j I_0}. \tag{4}$$

### 3.1 Global Gaussian moment enclosures

For every $t \ge 0$, Taylor's theorem and its integral remainder give

$$0 \le R(t) - \frac{t^2}{2} - \frac{t^3}{6} \le \frac{t^4 e^t}{24}, \qquad R(t) \le \frac{t^2 e^t}{2}.$$

Also $0 \le e^{-u} - 1 + u \le u^2/2$ for $u \ge 0$. Apply the last inequality to $u = R(t)$ to obtain the global bounds

$$-\frac{t^4 e^t}{24} \le e^{-R(t)} - 1 + \frac{t^2}{2} + \frac{t^3}{6} \le \frac{t^4 e^{2t}}{8}. \tag{5}$$

All of these inequalities hold for arbitrarily large $t$.

For $j = 0, 1, 2, 3$, the remainder integrals are explicit Gaussian moments. In particular

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} x^{2j+8} e^{-(1-2\varepsilon)x^2}\, dx = a_{j+4}(1-2\varepsilon)^{-j-9/2}.$$

Since $\varepsilon \le 1/100$,

$$(1-2\varepsilon)^{-j-9/2} \le (50/49)^8 < 2 \qquad (0 \le j \le 3). \tag{6}$$

The same estimate with $1-\varepsilon$ is smaller. Multiplying (5) by $x^{2j}e^{-x^2}/\sqrt{\pi}$, putting $t = \varepsilon x^2$, and integrating gives

$$\begin{aligned} L_j &= a_j - \frac{\varepsilon^2}{2}a_{j+2} - \frac{\varepsilon^3}{6}a_{j+3} - \frac{\varepsilon^4}{12}a_{j+4} \le I_j, \\ I_j &\le U_j = a_j - \frac{\varepsilon^2}{2}a_{j+2} - \frac{\varepsilon^3}{6}a_{j+3} + \frac{\varepsilon^4}{4}a_{j+4}. \end{aligned} \tag{7}$$

Every $L_j$ in this range is positive. For transparency, their exact values and the corresponding upper bounds are

| $j$ | $L_j$ | $U_j$ |
|---|---|---|
| 0 | 12799995195993/12800000000000 | 12799995196021/12800000000000 |
| 1 | 12799975971937/25600000000000 | 12799975972189/25600000000000 |
| 2 | 38399831747307/51200000000000 | 38399831750079/51200000000000 |
| 3 | 191998485218991/102400000000000 | 191998485255027/102400000000000 |

These rational evaluations follow directly from (3) and (7).

### 3.2 A negative sixth cumulant

Symmetry makes the sixth cumulant of the normalized density equal to

$$\kappa_6 = \mu_6 - 15\mu_4\mu_2 + 30\mu_2^3.$$

Its scaled numerator is

$$N = I_3 I_0^2 - 15 I_2 I_1 I_0 + 30 I_1^3, \qquad \kappa_6 = \frac{N}{C^3 I_0^3}.$$

All the moment bounds are positive, so exact rational substitution gives

$$\begin{aligned} N &\le U_3 U_0^2 - 15 L_2 L_1 L_0 + 30 U_1^3 \\ &= -\frac{5515478129986823923483314031707}{41943040000000000000000000000000000000000} < 0. \end{aligned} \tag{8}$$

Since the numerator upper bound is negative and $I_0 \le U_0$, the larger denominator $C^3 U_0^3$ gives a valid upper bound for the quotient. Consequently

$$\kappa_6 \le -\frac{5515478129986823923483314031707}{4194299277498256246631316955739704634522000000000} < -10^{-18}. \tag{9}$$

The accompanying `verify_cumulant.py` reproduces the rational bounds and these strict signs using integer and rational arithmetic. The all-real-domain enclosure is the analytic argument (5)–(7), rather than a finite sampling assertion.

### 3.3 The classical real-zero obstruction

The transform $F$ is real and even, $F(0) > 0$, and it has order one as established above. If all its zeros were real, its even Hadamard product could be paired as

$$\frac{F(z)}{F(0)} = \prod_{r_j>0} \left(1 - \frac{z^2}{r_j^2}\right), \qquad \sum_j r_j^{-2} < \infty, \tag{10}$$

where positive zeros are counted with multiplicity. The possible linear exponential factor is constant by evenness, and $F(0) > 0$ excludes a zero at the origin. Formula (10) is also valid when the set of zeros is finite or empty.

Near $t = 0$ the normalized moment generating function is $F(-it)/F(0)$. Absolute local convergence of the paired product gives

$$\log \frac{F(-it)}{F(0)} = \sum_j \log \left(1 + \frac{t^2}{r_j^2}\right).$$

Its $t^6$ coefficient is $\frac{1}{3}\sum_j r_j^{-6} \geq 0$. On the other hand that coefficient is $\kappa_6/6!$, negative by (9). This contradiction proves the existence of a nonreal zero. The cumulant sign implication is a classical consequence of the real-zero canonical product; the explicit moment enclosure above is what violates it for this kernel.

Finally, for every real $y$,

$$F(iy) = \int_{\mathbb{R}} \phi(s) \cosh(ys)\, ds > 0.$$

Thus a nonreal zero cannot lie on the imaginary axis. Reality and evenness produce a quartet $z, -z, \overline{z}, -\overline{z}$, all off both axes.

## 4 Gaussian localization and higher log-jet obstructions

The certificate above uses one low-order necessary sign. The following general calculation explains its large-damping origin. Its ingredients are classical Gaussian completion of the square, derivative convergence under approximate identities, and canonical products; it is a self-contained diagnostic alongside Newman's classification [1].

**Theorem 4.1.** *Fix an integer $n \geq 3$. Let $h : \mathbb{R} \to (0, \infty)$ be $C^n$, with $h$ and its derivatives through $n$ bounded. For $\alpha > 0$ let $p_\alpha(s)$ be the probability density proportional to $e^{-\alpha s^2} h(s)$, and let $\kappa_n(\alpha)$ be its $n$th cumulant. Put*

$$H_\alpha(v) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-x^2} h(v + x/\sqrt{\alpha})\, dx.$$

*Then*

$$\kappa_n(\alpha) = (2\alpha)^{-n} (\log H_\alpha)^{(n)}(0), \qquad \lim_{\alpha\to\infty} (2\alpha)^n \kappa_n(\alpha) = (\log h)^{(n)}(0). \tag{11}$$

*If $h$ is even and $(-1)^{m+1}(\log h)^{(2m)}(0) < 0$ at some fixed integer $m \geq 2$ with $2m \leq n$, its Gaussian-damped Fourier transform has nonreal zeros for every sufficiently large $\alpha$.*

*Proof.* For real $t$, completing the square and making a real change of variables gives

$$Z_\alpha(t) = \int_{\mathbb{R}} e^{-\alpha s^2 + ts} h(s)\, ds = \sqrt{\frac{\pi}{\alpha}}\, e^{t^2/(4\alpha)} H_\alpha(t/(2\alpha)).$$

Thus the logarithm of the normalized moment generating function is

$$\frac{t^2}{4\alpha} + \log H_\alpha(t/(2\alpha)) - \log H_\alpha(0).$$

Differentiation proves the first identity for $n \geq 3$. No evaluation of $h$ at complex arguments or contour displacement is involved. The cumulant formula applies for $n \geq 3$.

Bounded derivatives justify differentiation under the Gaussian integral. Dominated convergence gives $H_\alpha^{(j)}(0) \to h^{(j)}(0)$ for $0 \leq j \leq n$. Since $h(0) > 0$, the finite partition formula for logarithmic derivatives is continuous at this limiting vector of jets. This proves the limit in (11).

The Fourier transform is entire of order at most two and finite type, because $h$ is bounded. If all its zeros are real, pairing them as $\pm r_j$ in its Hadamard product gives for $m \geq 2$

$$\kappa_{2m}(\alpha) = (-1)^{m+1} \frac{(2m)!}{m} \sum_j r_j^{-2m}. \tag{12}$$

The paired canonical factors have only quadratic convergence corrections, and the even exponential polynomial also has degree at most two. Neither affects degree $2m \geq 4$. The displayed series converges because the zero exponent of convergence is at most two. The zero-free case has an empty sum. Hence $(-1)^{m+1}\kappa_{2m} \geq 0$ is necessary, whereas the limit in (11) gives the contrary sign for all sufficiently large $\alpha$ under the hypothesis. □

**Proposition 4.2.** *Under the preceding assumptions, suppose the derivatives through $n+2$ are bounded, and choose*

$$K \geq \max\{1, h(0)^{-1}, \max_{0 \leq j \leq n+2} \|h^{(j)}\|_\infty\}, \quad C_n(K) = n^n n! 2^{n-1} K^{2n+2}.$$

*For $\alpha \geq K^2$,*

$$\left| \kappa_n(\alpha) - \frac{(\log h)^{(n)}(0)}{(2\alpha)^n} \right| \leq \frac{C_n(K)}{2^n \alpha^{n+1}}.$$

*Proof.* Taylor's theorem, Gaussian mean zero and variance $1/2$ give $|H_\alpha^{(j)}(0) - h^{(j)}(0)| \leq \delta = K/(4\alpha)$ for $0 \leq j \leq n$. Thus $H_\alpha(0) \geq h(0)/2$. For $j \geq 1$, let $q_j = H_\alpha^{(j)}(0)/H_\alpha(0)$ and $r_j = h^{(j)}(0)/h(0)$. Both are bounded in absolute value by $R = 2K^2$, and

$$|q_j - r_j| \leq 2K\delta + 2K^3\delta \leq 4K^3\delta.$$

In the logarithmic-derivative formula, a set partition with $k$ blocks contributes $(k-1)!$ times a product of $k$ such ratios, with sign $(-1)^{k-1}$. Telescoping the product bounds its difference by $kR^{k-1}4K^3\delta$. There are at most $n^n$ set partitions and $k! \leq n!$. Consequently

$$|(\log H_\alpha)^{(n)}(0) - (\log h)^{(n)}(0)| \leq n^n n! R^{n-1} 4K^3\delta = \frac{C_n(K)}{\alpha}.$$

Combine this with (11). □

For $\psi(s) = \exp(-Ae^{s^2})$, write

$$e^{-Ms^2}\psi(s) = e^{-A}e^{-(M+A)s^2}h(s), \qquad h(s) = \exp[-A(e^{s^2} - 1 - s^2)].$$

This $h$ and every fixed derivative are bounded. Setting $\alpha = M + A$ in Theorem 4.1 gives, for each fixed $m \geq 2$,

$$\kappa_{2m}(M) \sim -\frac{A(2m)!}{m!\,2^{2m}(M+A)^{2m}}. \tag{13}$$

For every fixed odd $m \geq 3$ this is the forbidden sign in (12). Thus every such fixed cumulant order detects nonreal zeros throughout a sufficiently large damping tail. The threshold may depend on $m$. At $m = 3$ the leading term is $-15A/[8(M + A)^6]$; the preceding exact certificate supplies a numerical parameter without relying on an unspecified asymptotic threshold.

## 5 Consequences and limits

Theorem 1.1 rules out the proposed implication from full admissibility, ordinary uniform log-concavity and the *entire* positive uniformly log-concave continuous root hierarchy to real Fourier zeros. The mechanism is exact: quadratic damping is invisible to every logarithmic curvature in Lemma 2.1, while it can force a forbidden Fourier cumulant. Increasing that root-iteration depth therefore cannot supply the missing Fourier conclusion for this class.

This construction and the preceding finite-depth construction [4, 5] have distinct additional strengths. Here one kernel works at all root depths. Those earlier constructions can prescribe finite global generalized-Laguerre cutoffs, positive physical moment forms on an initial parameter interval, and a strip containing *all* Fourier zeros. None of those three conclusions is proved for the present kernel. In particular, one may dilate a selected nonreal zero toward the real axis, but that observation does not confine all zeros to a thin strip.

The kernel is a deliberately chosen analytic model, not the arithmetic Riemann theta density. Its continuous root hierarchy is also different from coefficientwise Turán iteration, the generalized Fourier-Laguerre index, and all-order positivity of the physical theta functional. A more restrictive criterion using additional structure may remain possible; the theorem specifies an input hierarchy that is insufficient by itself.

**Reproducibility and disclosure.** The accompanying standard-library Python file reconstructs the exact Gaussian-moment certificate without supplied moment tables or floating-point quadrature. Its arithmetic checks support the whole-line analytic inequalities and induction above. Research and manuscript preparation used AI assistance.

## References


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