Log-concave polynomial prefix sums: algebraic certificates and a sharp zero sector

Scott Schubr
Luminary Oddities

November 26, 2026
Archive edition 6

Abstract

We give algebraic certificates for joint logarithmic concavity of polynomial prefix sums $H_k(y) = \sum_{i=0}^{k} \rho^i h(y + ai)$, for every real step $a \geq 0$ and bias $\rho \geq 1$. It suffices that the factorial coefficients of h are nonnegative and log-concave with interval support, that $h(0) > 0$, and that $h(U)h(V)$ has a nonnegative expansion in UV and $U + V$. The resulting class is closed under products, positive affine changes of variable and nonzero derivatives. It contains all real polynomials with zeros in the closed sector $|\arg(-z)| \leq \pi/4$ and strictly more, including an explicit cubic outside that sector. The logarithmic extension agrees exactly at integer prefix lengths and gives Jensen inequalities for arbitrary finite-first-moment laws with integral mean prefix length. The sector angle and universal bias range are sharp. The analytic ingredient is a real two-variable truncated-square kernel, proved by complete chamber formulas and exact rational polynomial certificates. Classical Prékopa marginalization gives its sharp real smoothing threshold. An application covers split projective-line bundles over projective space in every dimension: every lattice polytope of a nef divisor on such a bundle has a log-concave sequence of Ehrhart evaluations, which answers a question of Ferroni and Higashitani for these polytopes (smooth for ample divisors), extending the unimodular twists covered by Leake and Mohammadi Yekta and the planar case to every twist and dimension. A separate kernel extension settles dimension three at radius two, with the sharp independent real-exponent range.

Keywords. log-concavity; polynomial zeros; prefix sums; Jensen inequalities; sector criteria; Ehrhart polynomials.
2020 Mathematics Subject Classification. 26D15, 05A20, 52B20, 14M25.

1 Statement and relation to previous work

The logarithm of a polynomial prefix sum is initially defined only at integral prefix lengths. To apply Jensen's inequality jointly in prefix length and height, one needs a concave extension that agrees exactly with those rows. We obtain such an extension from finite algebraic certificates and a mixed adjacent-row inequality. The certificate class strictly includes a sharp universal zero sector; its closure properties generate further admissible inputs.

Fix a real polynomial h, a step $a \geq 0$ and a bias $\rho > 0$, and write

$$H_k(y) = \sum_{i=0}^{k} \rho^i h(y + ai), \qquad k \in \mathbb{Z}_{\geq 0}, \quad y \geq 0. \tag{1}$$

The parameters h, a, ρ remain fixed when k, y vary. We use the principal argument to define

$$\mathcal{S} = \{z \neq 0 : |\arg(-z)| \leq \pi/4\}.$$

Constants are allowed in the following theorem; their zero condition is vacuous.

Theorem 1.1 (Sharp universal sector criterion)**.** *Suppose $h \in \mathbb{R}[t]$, $h(0) > 0$, and every zero of h belongs to $\mathcal{S}$. If $a \geq 0$ and $\rho \geq 1$, then:*

1. *Every function $y \mapsto \log H_k(y)$ is concave on $[0, \infty)$.*
2. *For all $k \geq 1$ and $u, v \geq 0$,*

$$H_k\left(\frac{u+v}{2}\right)^2 \geq H_{k-1}(u) H_{k+1}(v). \tag{2}$$

3. *There is a finite concave function J on $[0, \infty)^2$ with $J(k, y) = \log H_k(y)$ for every integer $k \geq 0$ and every $y \geq 0$.*

Consequently, for any probability law with $K \in \mathbb{Z}_{\geq 0}$, $Y \geq 0$, $\mathbb{E}(K + Y) < \infty$, and $\mathbb{E}K = \bar{k} \in \mathbb{Z}_{\geq 0}$, one has

$$\mathbb{E} \log H_K(Y) \leq \log H_{\bar{k}}(\mathbb{E}Y). \tag{3}$$

The expectation on the left is finite. In particular, the conclusion applies to arbitrary finite nonnegative weights of total mass one, with integral weighted mean of the prefix lengths.

Theorem 1.2 (Algebraic prefix certificates)**.** *Let $h(t) = \sum_{j=0}^{d} A_j t^j / j!$ with $A_0 > 0$. Suppose that $(A_j)_{j=0}^{d}$ is nonnegative, log-concave and interval-supported, and that there is a polynomial $Q(P, S)$ with nonnegative coefficients such that*

$$h(U)h(V) = Q(UV, U + V). \tag{4}$$

Then all conclusions of Theorem 1.1 hold for h, for every $a \geq 0$ and $\rho \geq 1$. These are sufficient algebraic certificates, not a characterization of every polynomial with the prefix property.

The sector condition implies both certificates by Lemma 2.3. The class is strictly larger and has the closure properties and explicit outside-sector example proved in Proposition 2.4. The proof of the prefix criterion is shared: Section 2 gives the rows, Section 6 gives adjacent-row comparison from (4), and Section 7 gives exact contact and integrable Jensen.

Theorem 1.3 (Universal sharpness)**.** *No sector $|\arg(-z)| \leq \theta$ with $\theta > \pi/4$ can replace $\mathcal{S}$ in Theorem 1.1, even if only the finite Jensen conclusion at integral heights is requested. For each fixed $0 < \rho < 1$, the conclusion fails for some $a \in \mathbb{Z}_{\geq 0}$ and the fixed polynomial $h(t) = (t+1)(t+2)/2$, at $K = 1$ and heights $0, 1, 2$.*

The angle is optimal as a *universal sufficient zero sector*. The bias parameter is positive throughout. Affine inputs have additional positive-bias cases; the universal obstruction in Theorem 1.3 is quadratic.

The row ingredients belong to classical log-concavity theory: Walkup's ultra-log-concave convolution theorem, ordinary convolution preservation, and Gurvits's exponential-generating-function theorem are used explicitly below; see [1, 2, 3, 4]. Melbourne–Palafox-Castillo [5] give a stronger established unweighted power-moment estimate than the elementary bound used in our row argument. Partial-diagonal positivity followed by Abel summation has classical precedents, including

Johnson–Goldschmidt [6]. That method and continuous Prékopa marginalization [7] are classical inputs.

There is substantial lattice overlap with Speyer [8]. His weighted marginal Lemma 5.3 covers the unit-step, integral-root, integral-height positive-root-product prefix model, including arbitrary real exponential lattice weights. His difference-bound framework also covers the integer-argument truncated-power kernels appearing below. Section 9 makes these reductions explicit. Our proof addresses the real-argument joint kernel and independently varying real heights and steps, where a simple lattice rescaling introduces congruence restrictions. The contribution is the complete real-parameter prefix criterion, its broader finite algebraic input class, and the truncated-square smoothing mechanism.

For a concrete complex-root example, take $h(t) = (t+1)^2 + 1$, whose zeros $-1 \pm i$ lie on the sector boundary. The theorem applies, for example, with the real step $a = \sqrt{2}$ and bias $\rho = 3/2$, while height varies over the entire nonnegative half-line. The mixed-row inequality then joins these real heights to the exact concave extension in prefix length.

The geometric application answers a question on Ehrhart polynomials. Ferroni and Higashitani ask when the evaluations $E_P(0), E_P(1), E_P(2), \dots$ of the Ehrhart polynomial of a lattice polytope P form a log-concave sequence [9, Question 5.1]; their conjecture that every polytope with the integer decomposition property has this property [9, Conjecture 5.7] fails for smooth nine-dimensional polytopes [10, Theorem 1.3]. Let P be the lattice polytope of the nef divisor $\mathcal{O}(k_0) \otimes \pi^*\mathcal{O}(y_0)$, $k_0, y_0 \in \mathbb{Z}_{\geq 0}$, on the split bundle $\mathbb{P}_{\mathbb{P}^m}(\mathcal{O} \oplus \mathcal{O}(a))$, $m \geq 1$, $a \in \mathbb{Z}_{\geq 0}$. Then $E_P(k) = Z_{m,a,1}(kk_0, ky_0)$ in the notation of Corollary 8.1, and its logarithmic Jensen inequality for the law $\frac{1}{2}\delta_{((k-1)k_0,(k-1)y_0)} + \frac{1}{2}\delta_{((k+1)k_0,(k+1)y_0)}$, whose first coordinate has the integral mean kk_0, gives

$$E_P(k)^2 \geq E_P(k-1)\, E_P(k+1) \qquad (k \geq 1).$$

This answers the question positively for all these polytopes, in every dimension. For $m = 1$ they are lattice polygons, covered by [9, Theorem 5.9]; for $a \leq 1$ the fan of the bundle is unimodular, and the inequality also follows from the theorem of Leake and Mohammadi Yekta on unimodular polytopes [11, Corollary 1.5]. For $m \geq 2$ and $a \geq 2$ the fan contains $m+1$ rays spanning a sublattice of index a, and the ample polytopes form a family of smooth polytopes in every dimension $m+1 \geq 3$ with log-concave Ehrhart evaluations.

This sixth edition states the question of Ferroni and Higashitani and the part of it that Corollary 8.1 answers, and restates the scope of the radius-two extension positively; the theorems and proofs of the fifth edition are unchanged.

2 Factorial coefficients and every row

A nonnegative finite sequence has *interval support* if its positive indices form an interval. Log-concavity always includes this support condition. Write a polynomial as

$$h(t) = \sum_{n \geq 0} A_n \frac{t^n}{n!}.$$

We shall use three classical facts. Ordinary convolution preserves log-concavity of nonnegative interval-supported sequences [2, 4]. Binomial convolution $C_n = \sum_j \binom{n}{j} A_j B_{n-j}$ also preserves log-concavity (Walkup's ultra-log-concavity theorem applied to $A_n/n!$ and $B_n/n!$) [1, 4]. Finally a log-concave sequence A_n gives a log-concave exponential generating polynomial on the positive half-line [3, 4]. These statements also hold for reversed finite sequences and for their nonzero consecutive shifts.

Lemma 2.1 (Nondecreasing-weight moments)**.** *For $0 \le w_0 \le \cdots \le w_k$, not all zero, put $S_0 = \sum_{i=0}^{k} w_i$ and $S_l = \sum_{i=1}^{k} w_i i^l$ for $l \ge 1$. The sequence $C_l = S_l/l!$, $l \ge 0$, is log-concave with interval support.*

Proof. The case $k = 0$ is immediate. For $s > 0$ tilt $I \in \{1, \dots, k\}$ by probabilities proportional to $w_i i^s$, and set $X = \log(k/I)$. For $1 \le j \le k$, the injection $i \mapsto i + k - j$ gives

$$\sum_{i=1}^{j} w_i (i/j)^s \le \sum_{i=1}^{k} w_i (i/k)^s,$$

because $i/j \le (i + k - j)/k$ and $w_i \le w_{i+k-j}$. With $j = \lfloor ke^{-t} \rfloor$, this implies $\Pr(X \ge t) \le e^{-st}$. Thus

$$(\log S_s)'' = \operatorname{Var}(\log I) \le \mathbb{E}X^2 \le 2/s^2.$$

For integers $l \ge 3$, triangular integration on $[l - 1, l + 1]$ yields

$$\log \frac{S_{l-1} S_{l+1}}{S_l^2} \le 2 \log \frac{l^2}{l^2 - 1} \le \log \frac{l+1}{l}.$$

The last inequality follows from $l^4 - 2l^3 - 2l^2 + l + 1 \ge 0$: at $l = 3 + R$ it is $R^4 + 10R^3 + 34R^2 + 43R + 13$. For $l = 1$, normalize all weights on $\{0, \dots, k\}$. Pairing i and $k - i$ gives $\mu = \mathbb{E}I \ge k/2$, while $\nu = \mathbb{E}I^2 \le k\mu \le 2\mu^2$. Hence $2S_1^2 \ge S_0 S_2$, with the zero atom retained. For $l = 2$, $S_3 \le kS_2$. Chebyshev's covariance inequality under reference weights proportional to i gives

$$\frac{S_2}{S_1} \ge \frac{\sum_{i=1}^{k} i^2}{\sum_{i=1}^{k} i} = \frac{2k+1}{3} \ge \frac{2k}{3}.$$

Therefore $2S_1 S_3 \le 3S_2^2$. These are exactly the remaining factorial-normalized inequalities. Zeros and the support condition are immediate from the definitions. □

Lemma 2.2 (Row preservation)**.** *If A_n is nonnegative, log-concave and interval-supported, then $\sum_{i=0}^{k} w_i h(y + ai)$ has concave logarithm on its positive half-line for every $a \ge 0$ and every finite nondecreasing weight sequence as in Lemma 2.1.*

Proof. Its factorial coefficients are

$$B_q = \sum_{l \ge 0} A_{q+l} a^l \frac{S_l}{l!}.$$

Reverse the finite sequence A to recognize an ordinary convolution with the log-concave sequence $a^l C_l$; reverse back to obtain log-concavity of B. The exponential-generating-function theorem then gives row log-concavity. The case $a = 0$ is a scalar multiple of h. □

Lemma 2.3 (Sector input)**.** *Under the hypotheses of Theorem 1.1, h has positive coefficients through its degree, its factorial coefficient sequence is log-concave, and*

$$h(U)h(V) = \sum_{p,q \ge 0} c_{pq} (UV)^p (U + V)^q, \qquad c_{pq} \ge 0. \tag{5}$$

Proof. Up to a positive scalar, factor h into positive affine factors and quadratics $g(t) = At^2 + Bt + C$ with $A, C > 0$, $B \ge 0$ and $B^2 \ge 2AC$. Indeed a conjugate root pair gives $(t + u)^2 + v^2$, where $u > 0$ and $|v| \le u$; a real root gives $t + c$ with $c > 0$. The scalar is positive because $h(0) > 0$. Each factor's factorial coefficients are log-concave: in the quadratic case the sole nontrivial inequality is

$B^2 \geq 2AC$. Walkup's binomial-convolution closure proves the assertion for their product. With $P = UV$, $S = U + V$, direct multiplication gives

$$g(U)g(V) = A^2P^2 + ABPS + ACS^2 + (B^2 - 2AC)P + BCS + C^2. \tag{6}$$

The corresponding affine identity is immediate. Multiplying these nonnegative polynomials in P, S proves (5). □

Lemma 2.2 with $w_i = \rho^i$ proves the first assertion of Theorem 1.1. In particular h itself is positive, increasing, and log-concave on $[0, \infty)$; if it is nonconstant, its logarithmic derivative is positive and nonincreasing.

Proposition 2.4 (Closure and an outside-sector input)**.** *The class in Theorem 1.2 is closed under products, positive scalar multiples, changes $h(t) \mapsto h(\lambda t + c)$ with $\lambda > 0, c \geq 0$, and nonzero derivatives. It includes*

$$h(t) = (t + 1)(10t^2 + 20t + 21) = 10t^3 + 30t^2 + 41t + 21,$$

whose nonreal zeros are outside the sector $\mathcal{S}$.

Proof. Products preserve factorial-coefficient log-concavity by Walkup's binomial convolution, and their Q polynomials multiply. Positive scalar multiples cause no change of signs. Under the positive affine change, the new factorial coefficients are

$$\widetilde{A}_j = \lambda^j \sum_{l \geq 0} A_{j+l} \frac{c^l}{l!}.$$

Reversing the finite A sequence identifies the inner sum with an ordinary convolution of log-concave interval-supported sequences. The factor λ^j preserves log-concavity. In the second certificate, substitute

$$P \mapsto \lambda^2 P + \lambda c S + c^2, \qquad S \mapsto \lambda S + 2c,$$

which preserves nonnegative coefficients.

For a nonconstant input, differentiation shifts the factorial sequence to A_{j+1}, preserving its properties and its positive constant term. Apply $\partial_U \partial_V$ to (4). For each monomial the identity

$$\partial_U \partial_V (P^p S^q) = p(p + q)P^{p-1}S^q + q(q - 1)P^p S^{q-2}, \qquad P = UV,\ S = U + V,$$

has nonnegative coefficients; terms with zero prefactors are omitted. It gives a nonnegative certificate for $h'(U)h'(V)$.

For the displayed cubic, the factorial coefficients are $(21, 41, 60, 60)$, with $41^2 \geq 21 \cdot 60$ and $60^2 \geq 41 \cdot 60$. Direct multiplication gives its second certificate:

$$Q(P, S) = 100P^3 + 300P^2S + 410PS^2 + 210S^3 + 80P^2 + 600PS + 421P + 630S^2 + 861S + 441.$$

All coefficients are positive. Its roots are -1 and $-1 \pm i\sqrt{11/10}$, so the two nonreal roots lie strictly outside $\mathcal{S}$. Thus the sector is a sharp universal sufficient condition, while individual polynomials beyond it can have the full prefix property. □

3 The real truncated-square kernel

For integer $r \ge 0$ and $p > 0$ define

$$T_{p,r}(A,B) = \sum_{\substack{I,J\in\mathbb{Z}_{\ge 0}\\ I+J\le r}} (A-I)_+^p (B-J)_+^p, \qquad x_+ = \max(x,0). \tag{7}$$

The distinction between integer r and real A, B matters.

Theorem 3.1 (Truncated-square smoothing)**.** *For every $r \in \mathbb{Z}_{\ge 0}$, $T_{2,r}$ is jointly log-concave on the positive quadrant. It extends by zero outside that quadrant as a log-concave nonnegative function.*

The proof has a finite exact polynomial component. We give all geometry cases and reconstruction formulas, together with the coefficient certificate in Table 1. The exact sparse-polynomial protocol in Section 10 reconstructs every coefficient using rational arithmetic. This portion is a computer-assisted identity proof, with unbounded parameter scope, by exact polynomial expansion. The script `verification/check_kernel_certificates.py`, included with this article, carries out that protocol and reproduces every count and sign in Table 1.

For a positive polynomial $f(t,u)$ put

$$\begin{aligned} X &= f_t^2 - f f_{tt}, \qquad Y = f_u^2 - f f_{uu}, \\ D &= f(f_{tt}f_{uu} - f_{tu}^2) - f_{tt}f_u^2 - f_{uu}f_t^2 + 2f_{tu}f_t f_u. \end{aligned} \tag{8}$$

Then

$$XY - (f_t f_u - f f_{tu})^2 = fD, \qquad \det \nabla^2 \log f = D/f^3. \tag{9}$$

Thus $X, Y, D \ge 0$ suffice. Multiplication of f by a positive constant does not affect the logarithmic conclusion.

Write $A = m + t, B = n + u$ with $m, n \in \mathbb{Z}_{\ge 0}$ and $0 < t, u < 1$, assume $m \le n$ by symmetry, and put $s = m + n - r$. Reversing indices gives

$$f = \sum_{\substack{0\le U\le m,\ 0\le V\le n\\ U+V\ge s}} (t+U)^2 (u+V)^2. \tag{10}$$

Use the power sums

$$\begin{aligned} S_0(l) &= l+1, & S_1(l) &= l(l+1)/2, \\ S_2(l) &= l(l+1)(2l+1)/6, & S_3(l) &= l^2(l+1)^2/4, \\ S_4(l) &= \frac{l(l+1)(2l+1)(3l^2+3l-1)}{30}, \\ S_5(l) &= \frac{l^2(l+1)^2(2l^2+2l-1)}{12}. \end{aligned}$$

and

$$F_l(x) = S_0(l)x^2 + 2S_1(l)x + S_2(l).$$

Product case: $s \le 0$. Here $f = F_m(t)F_n(u)$. For $F_l = \alpha x^2 + \beta x + \gamma$,

$$\beta^2 - 2\alpha\gamma = l(l-1)(l+1)^2/3 \ge 0 \quad (l \ge 1).$$

It follows that $F_l F_l'' - (F_l')^2 \le 0$ for $x \ge 0$; $F_0(x) = x^2$ is immediate. Hence this case is jointly log-concave.

Shallow case: $0 < s \le m$**.** The lower triangle removed from the rectangle is wholly contained in it, so

$$f = F_m(t)F_n(u) - \sum_{i=0}^{s-1}(t+i)^2 F_{s-1-i}(u). \tag{11}$$

For an explicit short reconstruction of the last sum, let $L_{ab}(s) = \sum_{i+j\le s-1} i^a j^b$, $0 \le a, b \le 2$. The symmetric table is

$$\begin{aligned}
&L_{00} = s(s+1)/2, && L_{01} = s(s-1)(s+1)/6,\\
&L_{02} = s^2(s-1)(s+1)/12, && L_{11} = s(s-2)(s-1)(s+1)/24,\\
&L_{12} = s(s-2)(s-1)(s+1)(2s-1)/120,\\
&L_{22} = s(s-2)(s-1)(s+1)(2s^2-2s+3)/360.
\end{aligned}$$

Thus the removed polynomial is $\sum_{a,b=0}^{2}\binom{2}{a}\binom{2}{b}t^{2-a}u^{2-b}L_{ab}(s)$. Set $s = R+1$, $m = s+M$, $n = s+N$. Expanding (8) in (R, M, N, t, u) gives the nonnegative coefficients in Table 1; the parameters are all nonnegative. The table identities also follow directly from Faulhaber's formulas, or by adding the new diagonal when s increases.

Middle case: $m < s \le n$**.** Write $L = s - m \ge 1$ and $N = n - s \ge 0$. Then

$$\begin{aligned}
f = \sum_{U=0}^{m}(t+U)^2\{&(N+U+1)u^2 + 2u[S_1(n) - S_1(s-U-1)]\\
&+ S_2(n) - S_2(s-U-1)\}.
\end{aligned} \tag{12}$$

The summand is a polynomial of degree at most five in U. Replace U^j by $S_j(m)$ using the displayed identities. For $m \ge 1$ set $m = M+1$, $L = l+1$ and $n = m+L+N$; for $m = 0$ use the same formula separately. The resulting coefficient signs are listed in Table 1. In the latter case $f = t^2\sum_{V=L}^{L+N}(u+V)^2$; its zero boundary causes no division inside the positive unit cell.

Deep case: $n < s \le m+n$**.** Here $r < m \le n$, and all terms of the triangle are untruncated. For $r = 0$ the answer is A^2B^2. For $r \ge 1$, divide by the positive cardinality $(r+1)(r+2)/2$. The normalized polynomial is

$$\begin{aligned}
f = A^2B^2 &- \frac{2r}{3}(A^2B + AB^2) + \frac{r(r+1)}{6}(A^2+B^2)\\
&+ \frac{r(r-1)}{3}AB - \frac{r(r-1)(2r+1)}{30}(A+B)\\
&+ \frac{r(r-1)(2r^2+2r+3)}{180}.
\end{aligned} \tag{13}$$

It follows by summing the six moments of I, J of bidegree at most $(2,2)$ on $I+J \le r$. Set $r = R+1$, $A = r+x$, $B = r+y$. The positive-coefficient certificate in (R, x, y) covers even the larger domain $A, B \ge r$. In this row of the table the last column checks the full determinant numerator fD, rather than D.

Completion of Theorem 3.1. The cases exhaust all nonnegative m, n, r. The formulas, exact coefficient signs and (9) prove negative semidefiniteness of the log-Hessian on each open unit cell. The function is positive on the positive quadrant, since the term $I = J = 0$ is positive. Each squared

Case and nonnegative parameters	X	Y	determinant	kind
Shallow: (R, M, N, t, u)	1419	1419	5662	D
Middle, $m \geq 1$: (M, l, N, t, u)	1121	705	3306	D
Middle, $m = 0$: (l, N, t, u)	65	22	80	D
Deep: (R, x, y)	58	58	337	fD

Table 1: Numbers of nonzero monomials in exact rational expansions. Every listed coefficient is positive. Missing coefficients are zero. These are complete polynomial expansions, with no parameter cutoff.

positive part is C^1, so the logarithm is C^1 across the integer walls. Restriction to any line segment has nonincreasing derivative inside its finitely many cells and a continuous derivative at every crossing. A segment contained in a wall follows by a limit. Thus the logarithm is concave globally. Extension by zero is log-concave because its positive support is the convex positive quadrant and boundary limits are zero. □

Lemma 3.2 (Monotone fractional marginal)**.** *Let $f : \mathbb{R}^2 \to [0, \infty)$ be finite, measurable, coordinatewise nondecreasing and log-concave, and vanish if either coordinate is nonpositive. For $\alpha, \beta > 0$, the function*

$$\mathcal{I}_{\alpha,\beta} f(A, B) = \int_0^\infty \int_0^\infty f(A - s, B - t) s^{\alpha-1} t^{\beta-1} \, ds \, dt$$

is finite and log-concave. The same holds if one or both orders are zero, interpreted as omitting that integration.

Proof. When $\alpha \geq 1$, retain s and its log-concave power factor. When $0 < \alpha < 1$, substitute $s = u^{1/\alpha}$, so $s^{\alpha-1} ds = du/\alpha$. The coordinate map $A - u^{1/\alpha}$ is concave on $u \geq 0$. Treat the other order independently in the same way. A coordinatewise nondecreasing log-concave function composed with these concave coordinate maps is log-concave: the image of an interpolated point dominates the interpolated images coordinatewise, so monotonicity followed by log-concavity proves the defining inequality. Multiply by the remaining log-concave power factors and orthant indicators. Prékopa marginalization, Theorem 6 of [7], proves the claim. The original fibers are confined to $0 \leq s \leq A_+$, $0 \leq t \leq B_+$ and their integrand is bounded by $f(A, B) s^{\alpha-1} t^{\beta-1}$. Thus the integrals are finite; the transformed fibers are bounded as well. Omitting one coordinate integration gives the zero-order cases. □

Corollary 3.3 (Sharp real smoothing range)**.** *For real $p, q > 0$ define*

$$T_{p,q,r}(A, B) = \sum_{I+J \leq r} (A - I)_+^p (B - J)_+^q.$$

This function is jointly log-concave for every $r \in \mathbb{Z}_{\geq 0}$ if and only if $p, q \geq 2$. In particular, for every real $p \geq 2$,

$$T_{p,r}(x, x) \geq T_{p,r}(x - d, x + d) \quad (x \geq 0,\ 0 \leq d \leq x). \tag{14}$$

Proof. The zero extension of $T_{2,r}$ is coordinatewise nondecreasing. For every real $p > 2$, the beta integral gives

$$\int_0^{z_+} (z - s)_+^2 s^{p-3} \, ds = B(3, p - 2) z_+^p = \frac{2\Gamma(p - 2)}{\Gamma(p + 1)} z_+^p.$$

Apply Lemma 3.2 with independent orders $p-2$ and $q-2$, omitting an integration when the corresponding exponent is two. Tonelli's theorem and the displayed identity yield a positive constant times $T_{p,q,r}$, proving sufficiency throughout the full real range. This uses coordinatewise monotonicity to cover orders below one, where the original power kernel is log-convex. For $p=q$, symmetry and midpoint log-concavity imply (14); the zero boundary follows continuously.

For necessity, take $r=1$ and $B=1$. For any $q>0$ the remaining row is $F_p(A)=A_+^p+(A-1)_+^p$. At $A=1-\varepsilon,1,1+\varepsilon$, its endpoint product minus its midpoint square is

$$(1-\varepsilon^2)^p+\varepsilon^p(1-\varepsilon)^p-1.$$

For $0<p<2$, division by ε^p gives a limit of one as $\varepsilon\downarrow 0$, since $(1-\varepsilon^2)^p-1=-p\varepsilon^2+o(\varepsilon^2)$. It is therefore positive for sufficiently small ε, a genuine midpoint violation. Interchanging coordinates treats $0<q<2$. For $p=1$, $\varepsilon=1/4$ gives the exact endpoint product $9/8>1$. This necessity is uniform over radii; radius zero alone is log-concave for all positive exponents. □

4 A three-dimensional radius-two extension

The same smoothing question has a bounded higher-dimensional extension. This section treats dimension three at radius two.

Theorem 4.1. *For positive real exponents p_1,p_2,p_3, define*

$$T_{p_1,p_2,p_3}(A,B,C)=\sum_{\substack{i,j,k\ge 0\\ i+j+k\le 2}}(A-i)_+^{p_1}(B-j)_+^{p_2}(C-k)_+^{p_3}.$$

This function, with its zero values outside the positive orthant, is jointly log-concave if and only if $p_1,p_2,p_3\ge 2$.

Proof. First consider three squared powers. If $0<A\le 1$, every term with $i\ge 1$ vanishes, leaving $A^2\sum_{j+k\le 2}(B-j)_+^2(C-k)_+^2$. Thus Corollary 3.3 proves log-concavity on this slab, and similarly for the other coordinates.

We give the deep-region input explicitly. For integer $r\ge 1$ and $A_j\ge r$, no positive-part truncation is needed in the three-variable radius-r sum. Normalize by $\binom{r+3}{3}$ and write f for the result. Its moments have the complete finite formula

$$M_a(r)=\sum_{0\le l_j\le a_j}\frac{6(r)_{l_1+l_2+l_3}}{(l_1+l_2+l_3+3)!}\prod_{j=1}^{3}\left\{\begin{matrix}a_j\\ l_j\end{matrix}\right\}l_j!,\qquad a_j\in\{0,1,2\},\tag{15}$$

where $(r)_l$ is a falling factorial and the braces are Stirling numbers of the second kind. Indeed

$$\sum_{i_1+i_2+i_3\le r}\prod_j\binom{i_j}{l_j}=\binom{r+3}{l_1+l_2+l_3+3}$$

by the generating function $t^{l_1+l_2+l_3}/(1-t)^{l_1+l_2+l_3+4}$, including the slack coordinate. Dividing by $\binom{r+3}{3}$ and converting powers to falling factorials proves (15). The polynomial is therefore specified without any coefficient data by

$$f(A)=\sum_{a\in\{0,1,2\}^3}(-1)^{|a|}\prod_j\binom{2}{a_j}A_j^{2-a_j}\,M_a(r).$$

Set $r = R + 1$ and $A = (r + x, r + y, r + z)$. Form $N_{ij} = f_i f_j - f f_{ij}$, differentiating in x, y, z only, with R held fixed. Complete rational expansion in (R, x, y, z) gives 108 nonzero terms in f, 446 in each diagonal entry of N, 4837 in each two-index principal minor, and 21263 in $\det N$. Every nonzero coefficient is positive; absent coefficients vanish. For an explicit reconstruction protocol, use (15) for its 27 moments, perform the printed substitution, differentiate the exact polynomial dictionaries, and expand each of the seven principal determinants by its permutation formula. Reject any nonpositive stored coefficient or an incorrect complete term count. This is the same finite rational arithmetic defined in Section 10, now in four variables. It has no radius cutoff. All seven principal minors are therefore nonnegative on the full nonnegative parameter orthant. Since the origin lattice term gives $f > 0$, $N = -f^2 \nabla^2 \log f$ proves log-concavity throughout the convex deep region. In particular this covers $A, B, C \geq 2$ at $r = 2$.

Only three mixed open types remain, up to coordinate permutation: 111, 112 and 122, where a coordinate of type one belongs to $(1, 2)$ and a coordinate of type two belongs to $(2, \infty)$. For $m \in \{(1,1,1), (1,1,2), (1,2,2)\}$ set $A = m + (x, y, z)$ and form

$$f_m(x) = \sum_{\substack{I \in \{0,1,2\}^3 \\ |I| \leq 2,\ I_j \leq m_j}} \prod_{j=1}^{3} (x_j + m_j - I_j)^2. \tag{16}$$

A type-one coordinate x_j lies in $(0, 1)$; a type-two coordinate is positive and unbounded. No index exceeds two, so there are no further chambers at large coordinates.

For each of these three polynomials form $N_{ij} = (f_m)_i (f_m)_j - f_m (f_m)_{ij}$ and each of its seven nonempty principal minors P. Let $d_j = \deg_{x_j} P$. For every bounded coordinate substitute $x_j = t_j/(1 + t_j)$ and multiply by $(1 + t_j)^{d_j}$; leave unbounded coordinates unchanged. A monomial $c_\alpha x^\alpha$ contributes

$$c_\alpha \prod_{j:\, m_j = 1} t_j^{\alpha_j} (1 + t_j)^{d_j - \alpha_j} \prod_{j:\, m_j = 2} x_j^{\alpha_j}.$$

Thus its coefficient at a bounded exponent β_j is multiplied by $\binom{d_j - \alpha_j}{\beta_j - \alpha_j}$, interpreted as zero outside its range. This gives a complete integer-coefficient prescription for all 21 transformed principal-minor polynomials. Their positive nonzero coefficient counts, in principal-index order, are

m	1	2	3	12	13	23	123
111	59	59	59	380	380	380	1192
112	71	71	75	429	437	437	1319
122	75	75	75	441	441	441	1331

Every coefficient is obtained from (16), exact differentiation, the principal determinant and the displayed binomial transform. All nonzero coefficients are positive; missing ones are zero. Each denominator-clearing factor is positive, so these finite exact certificates prove $N \succeq 0$ throughout every mixed chamber. They are coefficient identities over a full parameter domain, not numerical Hessian samples.

The low slabs, deep region and these mixed chambers exhaust the positive orthant. Squared positive parts are C^1, so the finite sum and its positive logarithm are C^1 at the walls $A_j = 1, 2$. On a line segment, the logarithmic derivative is nonincreasing on each of its finitely many chamber pieces and continuous at their boundaries. Wall-contained segments follow by continuity. Hence the logarithm is globally concave. Every coordinate boundary has zero limit; the positive support is convex, so extension by zero is log-concave without dividing at zero.

The squared kernel is coordinatewise nondecreasing. The fractional marginal argument of Lemma 3.2 works in three variables one coordinate at a time: for integration order at least one retain the log-concave power density; for order $0 < \alpha < 1$, substitute $s = u^{1/\alpha}$, obtaining constant measure and the concave coordinate map $A - u^{1/\alpha}$. Coordinatewise monotonicity followed by log-concavity proves that the composed integrand is log-concave, and Prékopa marginalization applies. Fibers remain bounded. The beta identity printed in Corollary 3.3 and Tonelli's theorem therefore raise the three exponents independently to all $p_j \geq 2$.

For necessity, suppose $0 < p_1 < 2$ and set $B = C = 1$. All terms with positive second or third index vanish. At $A = 1 - \varepsilon, 1, 1 + \varepsilon$, $0 < \varepsilon < 1$, the translate by two also vanishes. The endpoint product minus the midpoint square is exactly

$$(1 - \varepsilon^2)^{p_1} + \varepsilon^{p_1}(1 - \varepsilon)^{p_1} - 1.$$

After division by ε^{p_1} this tends to one, and hence is positive for small ε. This is a midpoint violation for the fixed radius-two kernel. Permuting coordinates proves necessity for each exponent. It completes the theorem. □

The deep-region calculation and the two-dimensional reduction are separate from the new mixed-chamber coefficient calculation. Higher three-dimensional radii and arbitrary dimension at radius at least two are not settled by this theorem.

5 All rectangular moment sectors

Fix $k \geq 1$, $b \geq 0$ and $0 \leq t \leq 1$. The target square is $0 \leq i, j \leq k$, with arguments $A = b + t + i$, $B = b + t + j$. The source rectangle is $0 \leq i \leq k - 1$, $0 \leq j \leq k + 1$, with arguments $A = b + 2t + i$, $B = b + j$. For a diagonal $i + j = s$, define

$$D_s^{(p)}(b, t) = \sum_{\substack{i+j=s \\ 0 \leq i,j \leq k}} [(b + t + i)(b + t + j)]^p - \sum_{\substack{i+j=s \\ 0 \leq i \leq k-1,\ 0 \leq j \leq k+1}} [(b + 2t + i)(b + j)]^p.$$

For $p = 0$ use $0^0 = 1$.

Lemma 5.1 (Every upper diagonal tail)**.** *For all integer $p \geq 0$, $b \geq 0$, $t \in [0, 1]$ and $S \in \{0, \ldots, 2k\}$,*

$$\sum_{s=S}^{2k} D_s^{(p)}(b, t) \geq 0. \tag{17}$$

Proof. For $p = 0$, $D_s^{(0)} = \mathbf{1}_{s=k}$. Direct finite summation for $p = 1$ gives

$$D_s^{(1)} = \begin{cases} (s + 1)t^2, & s < k, \\ b^2 + b(k + 2t) + (k + 1)t^2, & s = k, \\ (2k - s + 1)(1 - t)^2, & s > k. \end{cases} \tag{18}$$

Thus these cases hold even term by term.

For $p \geq 2$, first set $b = 0$ and $r = 2k - S$. Reindex the target by $I = k - i$, $J = k - j$. The omitted triangle terms with $I > k$ or $J > k$ vanish by the positive part, so its upper tail is exactly $T_{p,r}(k + t, k + t)$. Reindex the source by $I = k - 1 - i$, $J = k + 1 - j$. Its upper tail is at most $T_{p,r}(k - 1 + 2t, k + 1)$: this enlarges the allowed index set and adds only nonnegative terms. In particular, when $t > 1/2$ the extra terms with $I = k$ must not be silently discarded. Since the two

arguments have mean $k+t$ and are nonnegative, (14) proves the required upper-tail comparison. This treats *all* cutoffs, including the low cutoffs, at once.

For general b, induct on p. Differentiation gives

$$\partial_b D_s^{(p)} = p(2b+2t+s)D_s^{(p-1)}.$$

If a sequence d_s has all upper tails nonnegative and w_s is nonnegative and nondecreasing, Abel summation gives

$$\sum_{s=S}^{2k} w_s d_s = w_S \sum_{s=S}^{2k} d_s + \sum_{j=S+1}^{2k} (w_j - w_{j-1}) \sum_{s=j}^{2k} d_s \geq 0. \tag{19}$$

Apply this with $w_s = 2b+2t+s$ and the induction hypothesis. The derivative of each p-tail is nonnegative, and the value at $b=0$ was proved above. This completes the induction. □

Proposition 5.2 (Unbounded degree and geometric bias)**.** *For all integers $p, q \geq 0$ and $\rho \geq 1$, the target-minus-source sum of $(AB)^p(A+B)^q$, with the product weight ρ^{i+j} on each rectangle, is nonnegative.*

Proof. On diagonal s both rectangles have the same argument sum $A+B = 2b+2t+s$. The difference is therefore

$$\sum_{s=0}^{2k} \rho^s (2b+2t+s)^q D_s^{(p)}.$$

The prefactor is nonnegative and nondecreasing in s, including $q=0$, so Lemma 5.1 and (19) apply. There is no bound on p or q. □

6 From the middle strip to all heights

We now prove (2) under the two algebraic conditions of Theorem 1.2; Lemma 2.3 supplies them in Theorem 1.1. Constants are harmless; suppose first that h is nonconstant. Let $\sigma = h'/h$, which is nonincreasing by Section 2. The row score $(\log H_k)'$ is the $\rho^i h(y+ai)$-weighted mean of $\sigma(y+ai)$.

For $w = (u+v)/2$ and $\delta = (u-v)/2$, put

$$L(\delta) = \log H_{k-1}(w+\delta) + \log H_{k+1}(w-\delta), \quad -w \leq \delta \leq w.$$

Row concavity makes L concave. At $\delta = 0$, adding the two largest arguments to the shorter prefix lowers its score, hence $L'(0) \geq 0$. If $w \geq a$, the shorter list at $\delta = a$ is $w+a, \ldots, w+ka$. The longer list is $w-a, \ldots, w+ka$; on their common part the weights differ by one fixed factor ρ^2, while the two additional arguments are smaller and have larger scores. Therefore $L'(a) \leq 0$. Concavity shows that the maximum over the feasible interval is attained in $[0, \min(a,w)]$. Thus it suffices to prove the inequality when $0 \leq u-v \leq 2a$.

For $a > 0$, divide all arguments by a and replace $h(z)$ by $h(az)$, which preserves both algebraic conditions by positive rescaling. Write $u/a = b+2t$, $v/a = b$, $b \geq 0$, $0 \leq t \leq 1$. Expand $h(A)h(B)$ by (4). Each term is nonnegative by Proposition 5.2, with precisely the rectangle weights obtained from multiplying the two prefixes. This proves the middle strip, hence all heights.

For $a = 0$, $H_k(y) = G_k(\rho)h(y)$, where $G_k(\rho) = \sum_{i=0}^{k} \rho^i$ and

$$G_k(\rho)^2 - G_{k-1}(\rho)G_{k+1}(\rho) = \rho^k.$$

Row log-concavity of h gives the result directly. The same identity deals with a constant h, for every a. This completes the adjacent-row assertion for both prefix theorems.

7 Concave contact and integrable Jensen

Assume first that h is nonconstant. For $q > 0$ define

$$M_k(q) = \sup_{y \geq 0} H_k(y) e^{-qy}, \qquad m_k(q) = \log M_k(q).$$

The supremum is finite and attained because H_k is polynomial in y. The mixed inequality with independent second coordinates implies $M_{k-1}(q) M_{k+1}(q) \leq M_k(q)^2$, by taking separate maximizers. Thus $m_k(q)$ is a concave sequence. Let ϕ_q be its piecewise-linear interpolation and put

$$J(x, y) = \inf_{q > 0} \{\phi_q(x) + qy\}, \qquad x, y \geq 0. \tag{20}$$

Each function inside the infimum is concave, so their infimum is concave. Since $M_k(q) \geq H_k(0) \geq h(0)$, $J \geq \log h(0)$. Choosing one fixed $q > 0$ proves finiteness above. At any integer row and height $y \geq 0$, the positive supporting slope $q = (\log H_k)'(y)$ makes that height a maximizer in M_k, including the right supporting slope at zero. This proves $J(k, y) = \log H_k(y)$; the reverse inequality is immediate from the supremum definition. If h is constant, use explicitly $J(x, y) = \log h(0) + \phi(x)$, where ϕ linearly interpolates $\log G_k(\rho)$. Equivalently exact contact follows by $q \downarrow 0$.

Finite concavity gives (3) for finite laws. For an arbitrary probability law, let $z = (\mathbb{E}K, \mathbb{E}Y)$ and let F be the smallest coordinate face of the quadrant containing z. Every coordinate of the random point corresponding to a zero coordinate of z vanishes almost surely, by nonnegativity. Hence the law is concentrated on F, and z is in its relative interior. A finite concave function has a supporting affine majorant at every relative-interior point of its convex domain (apply the supporting hyperplane theorem to its hypograph). Thus there is an affine L on F with $L(z) = J(z)$ and $J \leq L$ on F. The assertion on a zero-dimensional face is immediate. The majorant also bounds all boundary points: concavity along a segment from z to such a point and the interior supporting inequality imply that same bound there.

On the actual integer rows $J(K, Y) = \log H_K(Y)$ is measurable, since K is integer-valued and each row is continuous. The bounds $\log h(0) \leq J(K, Y) \leq L(K, Y)$ and finite first moments imply absolute integrability. Taking expectations gives $\mathbb{E}J(K, Y) \leq L(z) = J(z)$. Integral $\mathbb{E}K$ supplies exact row contact at z, proving (3) without a countable-support or bounded-height restriction. This completes Theorems 1.1 and 1.2.

8 Sharpness and a geometric application

Proof of Theorem 1.3. For a wider angle choose $v > u > 0$ with $\arctan(v/u)$ inside it. The zeros of $h_L(t) = (t + Lu)^2 + (Lv)^2$ lie in that wider sector. At $K = 0$ the bias and step disappear, and direct calculation gives

$$h_L(0) h_L(2) - h_L(1)^2 = 2L^2(v^2 - u^2) - 4Lu - 1 > 0$$

for sufficiently large L. Equally weighted heights $0, 2$ have integral mean 1, contradicting logarithmic Jensen.

For $0 < \rho < 1$, use $h(t) = (t + 1)(t + 2)/2$, $K = 1$, and write $Z(y) = h(y) + \rho h(y + a)$. Exact expansion gives

$$Z(0) Z(2) - Z(1)^2 = \frac{\rho(1 - \rho)a^2 - 5\rho(1 + \rho)a - 6(1 + \rho)^2}{2}. \tag{21}$$

Its leading coefficient is positive. Hence a sufficiently large integer a gives the claimed integral-height counterexample. □

Corollary 8.1 (All-dimensional split bundles)**.** *Fix $m \geq 1$, $a \in \mathbb{Z}_{\geq 0}$ and $\rho \geq 1$. For integral $k, y \geq 0$ let*

$$Z_{m,a,\rho}(k,y) = \sum_{i=0}^{k} \rho^i \binom{y+ai+m}{m}.$$

These functions satisfy logarithmic Jensen inequalities for arbitrary probability laws under the first-moment and integral-mean-prefix hypotheses of Theorem 1.1. For $\rho = 1$ they are section counts of the split bundle $\mathbb{P}_{\mathbb{P}^m}(\mathcal{O} \oplus \mathcal{O}(a))$, with the quotient convention for the tautological bundle, at $\mathcal{O}(k) \otimes \pi^*\mathcal{O}(y)$.

Proof. The polynomial $h(t) = \binom{t+m}{m} = \prod_{j=1}^{m}(t+j)/m!$ has negative real zeros. The theorem applies, including the real height continuation. The projective-bundle pushforward is $\mathrm{Sym}^k(\mathcal{O} \oplus \mathcal{O}(a)) = \bigoplus_{i=0}^{k} \mathcal{O}(ai)$; taking sections gives the sum [12]. For $\rho \neq 1$ the formula is an explicitly weighted sum of these multiplicities, rather than an unweighted count. □

The twist a, dimension m and bias are fixed in a Jensen law. When the mean height is also integral, both sides in the unweighted case refer to actual integral divisor parameters. The application concerns this split-projective-bundle family.

The logarithm has a natural role here. For each fixed $\rho > 1$, no positive power can satisfy Jensen for all prefix lengths even with $a = 0$, $y = 0$ and constant input: for any $\alpha > 0$,

$$\frac{G_{k-1}(\rho)^\alpha + G_{k+1}(\rho)^\alpha}{2G_k(\rho)^\alpha} \longrightarrow \frac{\rho^{-\alpha} + \rho^\alpha}{2} > 1.$$

The concave function J in (20) gives the logarithmic continuation with exact contact on integer-prefix rows.

9 Lattice overlap and reproducibility

For positive integer c_j and unit step, the unweighted product prefix $\sum_{i=0}^{k} \prod_j (y + i + c_j)$ is a difference-bound lattice marginal. Take integer anchors (x_0, x_1, x_2), states v, w_j and inequalities

$$x_1 \leq v \leq x_0, \qquad v \leq w_j \leq x_2 + c_j - 1.$$

At anchors $(0, -k, y)$ the fibers have the indicated size, with $v = -i$. All constraints are fixed differences; the simultaneous translation symmetry gives the required L-convex model. The weight $\exp[\lambda(x_2 - v)]$ is logmodular and invariant under common translation. Speyer's weighted marginal Lemma 5.3 [8] therefore covers these integral unit-step cases for all real λ, after removal of the harmless factor $e^{\lambda y}$. This does not contradict (21), which uses larger steps.

Likewise, for integer positive A, B and integer $p \geq 1$, $T_{p,r}(A, B)$ has a fixed difference-bound lift. Use anchors $(x_0, x_1, x_2) = (0, A-1, 1-B)$, states I, J', X_l, Y_l, and

$$I \geq x_0, \quad J' \leq x_0, \quad I - J' \leq r, \quad I \leq X_l \leq x_1, \quad x_2 \leq Y_l \leq J' \quad (1 \leq l \leq p).$$

With $J = -J'$, each feasible (I, J) contributes $(A - I)^p(B - J)^p$, precisely the positive-part sum. Thus the integer-argument kernel already lies within the lattice marginal framework. The continuous real-argument assertion Theorem 3.1 requires additional work; the $p = 1$ example after Corollary 3.3 shows that integer marginal concavity alone does not ensure continuous log-concavity of this particular interpolation. Clearing rational denominators changes the step lattice and does not remove that distinction.

The four chamber formulas and Table 1 specify the finite algebraic calculations. Section 10 gives their complete rational polynomial protocol; no external coefficient table or numerical eigenvalue is a premise. The analytic convolution, source enlargement, Abel induction, score reduction and contact construction are proved in the text rather than inferred from finite calculations.

The assertions concern finite polynomials. The anisotropic kernel threshold in Corollary 3.3 is uniform over radii. Its fractional lift uses classical composition and marginalization.

10 Complete rational coefficient protocol

The finite calculation for Theorem 3.1 uses no sampled parameter values. Here is a full specification of its arithmetic, inputs and outputs. A polynomial in d variables is a finite dictionary $e \mapsto c_e \in \mathbb{Q}$, $e \in \mathbb{N}^d$; omit zero coefficients. Addition collects equal exponent vectors, multiplication sends (e, f) to $e + f$, and differentiation in coordinate i sends c_e to $e_i c_e$ at $e - \mathbf{e}_i$. Integer powers are repeated multiplication starting at one. Thus all steps below are finite rational operations.

Construct the polynomial f in each row of Table 1 as follows.

1. Shallow: use (11), the displayed $L_{ab}(s)$, $s = R + 1$, $m = s + M$, $n = s + N$, and expand in (R, M, N, t, u). There is no remaining sum of variable length: the removed triangle is its explicit nine-term moment formula.

2. Middle: use (12), expand its summand in a temporary variable U, and replace every U^j by the printed $S_j(m)$, $0 \le j \le 5$. For $m \ge 1$ substitute $m = M + 1$, $L = l + 1$, $s = m + L$, $n = s + N$; for $m = 0$ set $m = 0$ and keep $L = l + 1$, $n = L + N$. Expand in the corresponding parameter list in the table. The temporary variable is eliminated completely before differentiation.

3. Deep: use (13) with $r = R + 1$, $A = r + x$, $B = r + y$, and expand in (R, x, y). This is the normalized polynomial; its omitted triangle cardinality is a positive constant for each fixed integer r.

For each f, differentiate only the two spatial variables (t, u or x, y), holding all geometry parameters fixed. Form X, Y, D exactly by (8). The following identity is an additional coefficient check, not a division operation:

$$XY - (f_t f_u - f f_{tu})^2 = fD.$$

For shallow and both middle cases, collect the dictionaries of X, Y, D; for the deep case collect those of X, Y, fD. Every stored coefficient is strictly positive, and the numbers of stored terms are respectively $(1419, 1419, 5662)$, $(1121, 705, 3306)$, $(65, 22, 80)$ and $(58, 58, 337)$. These are all coefficients, not a selected positive subset. In algorithmic form the final acceptance step is:

> For every case and each of its three specified polynomials, collect all rational monomial coefficients, delete exact zeros, compare the complete term count with its stated count, and reject if the dictionary is empty or any remaining coefficient is nonpositive. Also reject if the displayed determinant identity fails as a polynomial dictionary equality.

This is a complete finite prescription from the printed chamber formulas; an absent coefficient is zero. The product and radius-zero cases are proved separately in the text. No cutoff in m, n, r is imposed.

The power-sum formulas may be verified independently by $S_j(z) - S_j(z-1) = z^j$ and the values at $z = 0$, for $0 \le j \le 5$. Alternatively all shallow moments follow from

$$\sum_{i+j \le s-1} (i)_a (j)_b = a! b! \binom{s+1}{a+b+2},$$

where falling factorials convert to ordinary powers by the finite Stirling identities. The analogous identity with $s+1$ replaced by $r+2$ reconstructs the deep formula before division by $(r+1)(r+2)/2$. Both identities follow by multiplying the elementary power-series generating functions. Thus the printed closed formulas, the exact polynomial test and the analytic chamber/gluing argument jointly prove the unrestricted two-dimensional result.

Statements and Declarations

Funding. This research was self-funded.

Competing interests. The author owns Luminary Oddities and has no other relevant competing interests.

Data and code availability. All mathematical arguments and finite exact reconstruction prescriptions are contained in this article; the exact coefficient checker for Table 1 is included with it.

AI assistance. OpenAI Codex assisted with research, literature comparison, proof checking, exact algebra, verification code and manuscript preparation. Anthropic Claude identified the answered question, edited this edition, checked the argument and wrote the coefficient checker. The author takes responsibility for the mathematical claims, proofs and references.

References

[1] D. W. Walkup, *Pólya sequences, binomial convolution and the union of random sets*, J. Appl. Probab. **13** (1976), 76–85.

[2] S. G. Hoggar, *Chromatic polynomials and logarithmic concavity*, J. Combin. Theory Ser. B **16** (1974), 248–254.

[3] L. Gurvits, *On multivariate Newton-like inequalities*, in *Advances in Combinatorial Mathematics*, Springer, 2009, 61–78. arXiv:0812.3687.

[4] A. Marsiglietti and J. Melbourne, *Geometric and functional inequalities for log-concave probability sequences*, arXiv:2004.12005v4 (2023), especially Corollaries 3.3, 3.4 and 3.7. Full text.

[5] J. Melbourne and G. Palafox-Castillo, *A discrete complement of Lyapunov's inequality and its information theoretic consequences*, arXiv:2111.06997 (2021). arXiv:2111.06997.

[6] O. Johnson and C. Goldschmidt, *Preservation of log-concavity on summation*, ESAIM Probab. Stat. **10** (2006), 206–215. doi:10.1051/ps:2006008.

[7] A. Prékopa, *On logarithmic concave measures and functions*, Acta Sci. Math. (Szeged) **34** (1973), 335–343. Author copy.

[8] D. E. Speyer, *L-log-concavity and a proof of the conjecture of Lam, Postnikov and Pylyavskyy*, arXiv:2601.05007v2 (2026). arXiv:2601.05007.

[9] L. Ferroni and A. Higashitani, *Examples and counterexamples in Ehrhart theory*, EMS Surv. Math. Sci. (2024), published online 17 September 2024. doi:10.4171/emss/86; version read and numbering used: arXiv:2307.10852v4.

[10] L. Ferroni, *Unimodality shenanigans in Ehrhart theory*, arXiv:2609.10513v2 (2026). arXiv:2609.10513v2.

[11] J. Leake and M. Mohammadi Yekta, *Log-concavity and approximate counting for totally unimodular polytopes*, arXiv:2609.39917 (v1 30 September 2026, v2 24 November 2026). arXiv:2609.39917.

[12] The Stacks Project Authors, *The Stacks Project*, Section 27.21, Example 27.21.3. Tag 01OA.