# A Sharp Zero-Sector Criterion for Polynomial Prefix Sums

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**Abstract**

Let $h$ be a real polynomial with $h(0) > 0$ whose zeros lie in the closed sector $|\arg(-z)| \le \pi/4$. For every real step $a \ge 0$ and geometric bias $\rho \ge 1$, we prove that $H_k(y) = \sum_{i=0}^{k} \rho^i h(y + ai)$ admits a finite concave logarithmic extension from $k \in \mathbb{Z}_{\ge 0}$, $y \ge 0$ to the nonnegative quadrant. This implies weighted logarithmic Jensen inequalities for arbitrary probability laws with finite first moments and integral mean prefix length. The sector angle and the universal range $\rho \ge 1$ are sharp, including counterexamples with integral heights. The main analytic ingredient is joint log-concavity of a two-variable truncated-square lattice sum. Four complete geometry cases and exact rational polynomial certificates establish that ingredient; classical Prékopa marginalization then removes the degree bound and yields a sharp real smoothing threshold. The resulting application covers split projective-line bundles over projective space in every dimension. The conclusion concerns a concave extension of integer-prefix rows, rather than an ordinary polynomial continuation in the prefix variable.

## 1 Statement and relation to previous work

Fix a real polynomial $h$, a step $a \ge 0$ and a bias $\rho > 0$, and write

$$H_k(y) = \sum_{i=0}^{k} \rho^i h(y + ai), \qquad k \in \mathbb{Z}_{\ge 0}, \quad y \ge 0. \tag{1}$$

The parameters $h, a, \rho$ remain fixed when $k, y$ vary. We use the principal argument to define

$$\mathcal{S} = \{z \ne 0 : |\arg(-z)| \le \pi/4\}.$$

Constants are allowed in the following theorem; their zero condition is vacuous.

**Theorem 1.1** (Sharp universal sector criterion)**.** *Suppose $h \in \mathbb{R}[t]$, $h(0) > 0$, and every zero of $h$ belongs to $\mathcal{S}$. If $a \ge 0$ and $\rho \ge 1$, then:*

1. *Every function $y \mapsto \log H_k(y)$ is concave on $[0, \infty)$.*
2. *For all $k \ge 1$ and $u, v \ge 0$,*

$$H_k\left(\frac{u+v}{2}\right)^2 \ge H_{k-1}(u) H_{k+1}(v). \tag{2}$$

3. *There is a finite concave function $J$ on $[0, \infty)^2$ with $J(k, y) = \log H_k(y)$ for every integer $k \ge 0$ and every $y \ge 0$.*

*Consequently, for any probability law with $K \in \mathbb{Z}_{\geq 0}$, $Y \geq 0$, $\mathbb{E}(K+Y) < \infty$, and $\mathbb{E}K = \bar{k} \in \mathbb{Z}_{\geq 0}$, one has*

$$\mathbb{E} \log H_K(Y) \leq \log H_{\bar{k}}(\mathbb{E}Y). \tag{3}$$

*The expectation on the left is finite. In particular, the conclusion applies to arbitrary finite nonnegative weights of total mass one, with integral weighted mean of the prefix lengths.*

**Theorem 1.2** (Universal sharpness)**.** *No sector $|\arg(-z)| \leq \theta$ with $\theta > \pi/4$ can replace $\mathcal{S}$ in Theorem 1.1, even if only the finite Jensen conclusion at integral heights is requested. For each fixed $0 < \rho < 1$, the conclusion fails for some $a \in \mathbb{Z}_{\geq 0}$ and the fixed polynomial $h(t) = (t+1)(t+2)/2$, at $K = 1$ and heights $0, 1, 2$.*

The angle is optimal as a *universal sufficient zero sector*. The sharpness statement concerns the universal sufficient sector. Bias zero is excluded throughout. Affine inputs have additional positive-bias cases; the universal obstruction in Theorem 1.2 is quadratic.

The row ingredients belong to classical log-concavity theory: Walkup's ultra-log-concave convolution theorem, ordinary convolution preservation, and Gurvits's exponential-generating-function theorem are used explicitly below; see [1, 2, 3, 4]. Melbourne–Palafox-Castillo [5] give a stronger established unweighted power-moment estimate than the elementary bound used in our row argument. Partial-diagonal positivity followed by Abel summation has classical precedents, including Johnson–Goldschmidt [6]. That method and continuous Prékopa marginalization [7] are classical inputs.

There is substantial lattice overlap with Speyer [8]. His weighted marginal Lemma 5.3 covers the unit-step, integral-root, integral-height positive-root-product prefix model, including arbitrary real exponential lattice weights. His difference-bound framework also covers the integer-argument truncated-power kernels appearing below. Section 8 makes these reductions explicit. Our proof addresses the real-argument joint kernel and independently varying real heights and steps, where a simple lattice rescaling introduces congruence restrictions. The contribution is the complete real-parameter prefix criterion and the finite truncated-square smoothing mechanism. The earlier quadratic and threefold investigation [10] supplied the initial prefix problem. Its Fano classification and power-concavity results are separate from the theorem proved here.

## 2 Factorial coefficients and every row

A nonnegative finite sequence has *interval support* if its positive indices form an interval. Log-concavity always includes this support condition. Write a polynomial as

$$h(t) = \sum_{n \geq 0} A_n \frac{t^n}{n!}.$$

We shall use three classical facts. Ordinary convolution preserves log-concavity of nonnegative interval-supported sequences [2, 4]. Binomial convolution $C_n = \sum_j \binom{n}{j} A_j B_{n-j}$ also preserves log-concavity (Walkup's ultra-log-concavity theorem applied to $A_n/n!$ and $B_n/n!$) [1, 4]. Finally a log-concave sequence $A_n$ gives a log-concave exponential generating polynomial on the positive half-line [3, 4]. These statements also hold for reversed finite sequences and for their nonzero consecutive shifts.

**Lemma 2.1** (Nondecreasing-weight moments)**.** *For $0 \leq w_0 \leq \cdots \leq w_k$, not all zero, put $S_0 = \sum_{i=0}^{k} w_i$ and $S_l = \sum_{i=1}^{k} w_i i^l$ for $l \geq 1$. The sequence $C_l = S_l/l!$, $l \geq 0$, is log-concave with interval support.*

*Proof.* The case $k = 0$ is immediate. For $s > 0$ tilt $I \in \{1, \dots, k\}$ by probabilities proportional to $w_i i^s$, and set $X = \log(k/I)$. For $1 \le j \le k$, the injection $i \mapsto i + k - j$ gives

$$\sum_{i=1}^{j} w_i (i/j)^s \le \sum_{i=1}^{k} w_i (i/k)^s,$$

because $i/j \le (i + k - j)/k$ and $w_i \le w_{i+k-j}$. With $j = \lfloor ke^{-t} \rfloor$, this implies $\Pr(X \ge t) \le e^{-st}$. Thus

$$(\log S_s)'' = \operatorname{Var}(\log I) \le \mathbb{E}X^2 \le 2/s^2.$$

For integers $l \ge 3$, triangular integration on $[l - 1, l + 1]$ yields

$$\log \frac{S_{l-1} S_{l+1}}{S_l^2} \le 2 \log \frac{l^2}{l^2 - 1} \le \log \frac{l + 1}{l}.$$

The last inequality follows from $l^4 - 2l^3 - 2l^2 + l + 1 \ge 0$: at $l = 3 + R$ it is $R^4 + 10R^3 + 34R^2 + 43R + 13$. For $l = 1$, normalize all weights on $\{0, \dots, k\}$. Pairing $i$ and $k - i$ gives $\mu = \mathbb{E}I \ge k/2$, while $\nu = \mathbb{E}I^2 \le k\mu \le 2\mu^2$. Hence $2S_1^2 \ge S_0 S_2$, with the zero atom retained. For $l = 2$, $S_3 \le kS_2$. Chebyshev's covariance inequality under reference weights proportional to $i$ gives

$$\frac{S_2}{S_1} \ge \frac{\sum_{i=1}^{k} i^2}{\sum_{i=1}^{k} i} = \frac{2k + 1}{3} \ge \frac{2k}{3}.$$

Therefore $2S_1 S_3 \le 3S_2^2$. These are exactly the remaining factorial-normalized inequalities. Zeros and the support condition are immediate from the definitions. □

**Lemma 2.2** (Row preservation)**.** *If $A_n$ is nonnegative, log-concave and interval-supported, then $\sum_{i=0}^{k} w_i h(y + ai)$ has concave logarithm on its positive half-line for every $a \ge 0$ and every finite nondecreasing weight sequence as in Lemma 2.1.*

*Proof.* Its factorial coefficients are

$$B_q = \sum_{l \ge 0} A_{q+l} a^l \frac{S_l}{l!}.$$

Reverse the finite sequence $A$ to recognize an ordinary convolution with the log-concave sequence $a^l C_l$; reverse back to obtain log-concavity of $B$. The exponential-generating-function theorem then gives row log-concavity. The case $a = 0$ is a scalar multiple of $h$. □

**Lemma 2.3** (Sector input)**.** *Under the hypotheses of Theorem 1.1, $h$ has positive coefficients through its degree, its factorial coefficient sequence is log-concave, and*

$$h(U)h(V) = \sum_{p,q \ge 0} c_{pq} (UV)^p (U + V)^q, \qquad c_{pq} \ge 0. \tag{4}$$

*Proof.* Up to a positive scalar, factor $h$ into positive affine factors and quadratics $g(t) = At^2 + Bt + C$ with $A, C > 0$, $B \ge 0$ and $B^2 \ge 2AC$. Indeed a conjugate root pair gives $(t + u)^2 + v^2$, where $u > 0$ and $|v| \le u$; a real root gives $t + c$ with $c > 0$. The scalar is positive because $h(0) > 0$. Each factor's factorial coefficients are log-concave: in the quadratic case the sole nontrivial inequality is $B^2 \ge 2AC$. Walkup's binomial-convolution closure proves the assertion for their product. With $P = UV$, $S = U + V$, direct multiplication gives

$$\begin{aligned} g(U)g(V) = {} & A^2 P^2 + ABPS + ACS^2 \\ & + (B^2 - 2AC)P + BCS + C^2. \end{aligned} \tag{5}$$

The corresponding affine identity is immediate. Multiplying these nonnegative polynomials in $P, S$ proves (4). □

Lemma 2.2 with $w_i = \rho^i$ proves the first assertion of Theorem 1.1. In particular $h$ itself is positive, increasing, and log-concave on $[0, \infty)$; if it is nonconstant, its logarithmic derivative is positive and nonincreasing.

## 3 The real truncated-square kernel

For integer $r \geq 0$ and $p > 0$ define

$$T_{p,r}(A, B) = \sum_{\substack{I,J \in \mathbb{Z}_{\geq 0} \\ I+J \leq r}} (A - I)_+^p (B - J)_+^p, \qquad x_+ = \max(x, 0). \tag{6}$$

The distinction between integer $r$ and real $A, B$ matters.

**Theorem 3.1** (Truncated-square smoothing)**.** *For every $r \in \mathbb{Z}_{\geq 0}$, $T_{2,r}$ is jointly log-concave on the positive quadrant. It extends by zero outside that quadrant as a log-concave nonnegative function.*

The proof has a finite exact polynomial component. We give all geometry cases and reconstruction formulas, together with the coefficient certificate in Table 1. The accompanying standard-library checker `verify_prefix.py` reconstructs and checks every coefficient using rational arithmetic. This portion is a computer-assisted identity proof, with unbounded parameter scope; it is not a numerical or finite-grid inference.

For a positive polynomial $f(t, u)$ put

$$\begin{aligned} X &= f_t^2 - f f_{tt}, \qquad Y = f_u^2 - f f_{uu}, \\ D &= f(f_{tt} f_{uu} - f_{tu}^2) - f_{tt} f_u^2 - f_{uu} f_t^2 + 2 f_{tu} f_t f_u. \end{aligned} \tag{7}$$

Then

$$XY - (f_t f_u - f f_{tu})^2 = f D, \qquad \det \nabla^2 \log f = D / f^3. \tag{8}$$

Thus $X, Y, D \geq 0$ suffice. Multiplication of $f$ by a positive constant does not affect the logarithmic conclusion.

Write $A = m + t, B = n + u$ with $m, n \in \mathbb{Z}_{\geq 0}$ and $0 < t, u < 1$, assume $m \leq n$ by symmetry, and put $s = m + n - r$. Reversing indices gives

$$f = \sum_{\substack{0 \leq U \leq m,\ 0 \leq V \leq n \\ U+V \geq s}} (t + U)^2 (u + V)^2. \tag{9}$$

Use the power sums

$$\begin{aligned} S_0(l) &= l + 1, & S_1(l) &= l(l+1)/2, \\ S_2(l) &= l(l+1)(2l+1)/6, & S_3(l) &= l^2(l+1)^2/4, \\ S_4(l) &= \frac{l(l+1)(2l+1)(3l^2+3l-1)}{30}, \\ S_5(l) &= \frac{l^2(l+1)^2(2l^2+2l-1)}{12}. \end{aligned}$$

and

$$F_l(x) = S_0(l) x^2 + 2 S_1(l) x + S_2(l).$$

**Product case:** $s \le 0$**.** Here $f = F_m(t)F_n(u)$. For $F_l = \alpha x^2 + \beta x + \gamma$,

$$\beta^2 - 2\alpha\gamma = l(l-1)(l+1)^2/3 \ge 0 \quad (l \ge 1).$$

It follows that $F_l F_l'' - (F_l')^2 \le 0$ for $x \ge 0$; $F_0(x) = x^2$ is immediate. Hence this case is jointly log-concave.

**Shallow case:** $0 < s \le m$**.** The lower triangle removed from the rectangle is wholly contained in it, so

$$f = F_m(t)F_n(u) - \sum_{i=0}^{s-1} (t+i)^2 F_{s-1-i}(u). \tag{10}$$

For an explicit short reconstruction of the last sum, let $L_{ab}(s) = \sum_{i+j \le s-1} i^a j^b$, $0 \le a, b \le 2$. The symmetric table is

$$\begin{aligned}
&L_{00} = s(s+1)/2, && L_{01} = s(s-1)(s+1)/6,\\
&L_{02} = s^2(s-1)(s+1)/12, && L_{11} = s(s-2)(s-1)(s+1)/24,\\
&L_{12} = s(s-2)(s-1)(s+1)(2s-1)/120,\\
&L_{22} = s(s-2)(s-1)(s+1)(2s^2-2s+3)/360.
\end{aligned}$$

Thus the removed polynomial is $\sum_{a,b=0}^{2} \binom{2}{a}\binom{2}{b} t^{2-a}u^{2-b} L_{ab}(s)$. Set $s = R+1$, $m = s+M$, $n = s+N$. Expanding (7) in $(R, M, N, t, u)$ gives the nonnegative coefficients in Table 1; the parameters are all nonnegative. The table identities also follow directly from Faulhaber's formulas, or by adding the new diagonal when $s$ increases.

**Middle case:** $m < s \le n$**.** Write $L = s - m \ge 1$ and $N = n - s \ge 0$. Then

$$\begin{aligned}
f = \sum_{U=0}^{m} (t+U)^2 \{ &(N+U+1)u^2 + 2u[S_1(n) - S_1(s-U-1)] \\
&+ S_2(n) - S_2(s-U-1)\}.
\end{aligned} \tag{11}$$

The summand is a polynomial of degree at most five in $U$. Replace $U^j$ by $S_j(m)$ using the displayed identities. For $m \ge 1$ set $m = M+1$, $L = l+1$ and $n = m+L+N$; for $m = 0$ use the same formula separately. The resulting coefficient signs are listed in Table 1. In the latter case $f = t^2 \sum_{V=L}^{L+N} (u+V)^2$; its zero boundary causes no division inside the positive unit cell.

**Deep case:** $n < s \le m+n$**.** Here $r < m \le n$, and all terms of the triangle are untruncated. For $r = 0$ the answer is $A^2B^2$. For $r \ge 1$, divide by the positive cardinality $(r+1)(r+2)/2$. The normalized polynomial is

$$\begin{aligned}
f = A^2B^2 &- \frac{2r}{3}(A^2B + AB^2) + \frac{r(r+1)}{6}(A^2 + B^2)\\
&+ \frac{r(r-1)}{3}AB - \frac{r(r-1)(2r+1)}{30}(A+B)\\
&+ \frac{r(r-1)(2r^2+2r+3)}{180}.
\end{aligned} \tag{12}$$

It follows by summing the six moments of $I, J$ of bidegree at most $(2,2)$ on $I+J \le r$. Set $r = R+1$, $A = r+x$, $B = r+y$. The positive-coefficient certificate in $(R, x, y)$ covers even the larger domain $A, B \ge r$. In this row of the table the last column checks the full determinant numerator $fD$, rather than $D$.

| Case and nonnegative parameters | $X$ | $Y$ | determinant | kind |
|---|---|---|---|---|
| Shallow: $(R, M, N, t, u)$ | 1419 | 1419 | 5662 | $D$ |
| Middle, $m \geq 1$: $(M, l, N, t, u)$ | 1121 | 705 | 3306 | $D$ |
| Middle, $m = 0$: $(l, N, t, u)$ | 65 | 22 | 80 | $D$ |
| Deep: $(R, x, y)$ | 58 | 58 | 337 | $fD$ |

Table 1: Numbers of nonzero monomials in exact rational expansions. Every listed coefficient is positive. Missing coefficients are zero. These are complete polynomial expansions, with no parameter cutoff.

*Completion of Theorem 3.1.* The cases exhaust all nonnegative $m, n, r$. The formulas, exact coefficient signs and (8) prove negative semidefiniteness of the log-Hessian on each open unit cell. The function is positive on the positive quadrant, since the term $I = J = 0$ is positive. Each squared positive part is $C^1$, so the logarithm is $C^1$ across the integer walls. Restriction to any line segment has nonincreasing derivative inside its finitely many cells and a continuous derivative at every crossing. A segment contained in a wall follows by a limit. Thus the logarithm is concave globally. Extension by zero is log-concave because its positive support is the convex positive quadrant and boundary limits are zero. □

**Lemma 3.2** (Monotone fractional marginal)**.** *Let $f : \mathbb{R}^2 \to [0, \infty)$ be finite, measurable, coordinatewise nondecreasing and log-concave, and vanish if either coordinate is nonpositive. For $\alpha, \beta > 0$, the function*

$$\mathcal{I}_{\alpha,\beta} f(A, B) = \int_0^\infty \int_0^\infty f(A - s, B - t) s^{\alpha-1} t^{\beta-1} \, ds \, dt$$

*is finite and log-concave. The same holds if one or both orders are zero, interpreted as omitting that integration.*

*Proof.* When $\alpha \geq 1$, retain $s$ and its log-concave power factor. When $0 < \alpha < 1$, substitute $s = u^{1/\alpha}$, so $s^{\alpha-1} ds = du/\alpha$. The coordinate map $A - u^{1/\alpha}$ is concave on $u \geq 0$. Treat the other order independently in the same way. A coordinatewise nondecreasing log-concave function composed with these concave coordinate maps is log-concave: the image of an interpolated point dominates the interpolated images coordinatewise, so monotonicity followed by log-concavity proves the defining inequality. Multiply by the remaining log-concave power factors and orthant indicators. Prékopa marginalization, Theorem 6 of [7], proves the claim. The original fibers are confined to $0 \leq s \leq A_+$, $0 \leq t \leq B_+$ and their integrand is bounded by $f(A, B) s^{\alpha-1} t^{\beta-1}$. Thus the integrals are finite; the transformed fibers are bounded as well. Omitting one coordinate integration gives the zero-order cases. □

**Corollary 3.3** (Sharp real smoothing range)**.** *For real $p, q > 0$ define*

$$T_{p,q,r}(A, B) = \sum_{I+J \leq r} (A - I)_+^p (B - J)_+^q.$$

*This function is jointly log-concave for every $r \in \mathbb{Z}_{\geq 0}$ if and only if $p, q \geq 2$. In particular, for every real $p \geq 2$,*

$$T_{p,r}(x, x) \geq T_{p,r}(x - d, x + d) \quad (x \geq 0,\ 0 \leq d \leq x). \tag{13}$$

*Proof.* The zero extension of $T_{2,r}$ is coordinatewise nondecreasing. For every real $p > 2$, the beta integral gives

$$\int_0^{z_+} (z - s)_+^2 s^{p-3} \, ds = B(3, p - 2) z_+^p = \frac{2\Gamma(p - 2)}{\Gamma(p + 1)} z_+^p.$$

Apply Lemma 3.2 with independent orders $p-2$ and $q-2$, omitting an integration when the corresponding exponent is two. Tonelli's theorem and the displayed identity yield a positive constant times $T_{p,q,r}$, proving sufficiency throughout the full real range. This uses coordinatewise monotonicity to cover orders below one, where the original power kernel is log-convex. For $p=q$, symmetry and midpoint log-concavity imply (13); the zero boundary follows continuously.

For necessity, take $r=1$ and $B=1$. For any $q>0$ the remaining row is $F_p(A)=A_+^p+(A-1)_+^p$. At $A=1-\varepsilon,1,1+\varepsilon$, its endpoint product minus its midpoint square is

$$(1-\varepsilon^2)^p+\varepsilon^p(1-\varepsilon)^p-1.$$

For $0<p<2$, division by $\varepsilon^p$ gives a limit of one as $\varepsilon\downarrow 0$, since $(1-\varepsilon^2)^p-1=-p\varepsilon^2+o(\varepsilon^2)$. It is therefore positive for sufficiently small $\varepsilon$, a genuine midpoint violation. Interchanging coordinates treats $0<q<2$. For $p=1$, $\varepsilon=1/4$ gives the exact endpoint product $9/8>1$. This necessity is uniform over radii; radius zero alone is log-concave for all positive exponents. □

## 4 All rectangular moment sectors

Fix $k\ge 1$, $b\ge 0$ and $0\le t\le 1$. The target square is $0\le i,j\le k$, with arguments $A=b+t+i$, $B=b+t+j$. The source rectangle is $0\le i\le k-1$, $0\le j\le k+1$, with arguments $A=b+2t+i$, $B=b+j$. For a diagonal $i+j=s$, define

$$D_s^{(p)}(b,t)=\sum_{\substack{i+j=s\\0\le i,j\le k}}[(b+t+i)(b+t+j)]^p-\sum_{\substack{i+j=s\\0\le i\le k-1,\ 0\le j\le k+1}}[(b+2t+i)(b+j)]^p.$$

For $p=0$ use $0^0=1$.

**Lemma 4.1** (Every upper diagonal tail)**.** *For all integer $p\ge 0$, $b\ge 0$, $t\in[0,1]$ and $S\in\{0,\dots,2k\}$,*

$$\sum_{s=S}^{2k}D_s^{(p)}(b,t)\ge 0. \tag{14}$$

*Proof.* For $p=0$, $D_s^{(0)}=\mathbf{1}_{s=k}$. Direct finite summation for $p=1$ gives

$$D_s^{(1)}=\begin{cases}(s+1)t^2, & s<k,\\ b^2+b(k+2t)+(k+1)t^2, & s=k,\\ (2k-s+1)(1-t)^2, & s>k.\end{cases} \tag{15}$$

Thus these cases hold even term by term.

For $p\ge 2$, first set $b=0$ and $r=2k-S$. Reindex the target by $I=k-i$, $J=k-j$. The omitted triangle terms with $I>k$ or $J>k$ vanish by the positive part, so its upper tail is exactly $T_{p,r}(k+t,k+t)$. Reindex the source by $I=k-1-i$, $J=k+1-j$. Its upper tail is at most $T_{p,r}(k-1+2t,k+1)$: this enlarges the allowed index set and adds only nonnegative terms. In particular, when $t>1/2$ the extra terms with $I=k$ must not be silently discarded. Since the two arguments have mean $k+t$ and are nonnegative, (13) proves the required upper-tail comparison. This treats *all* cutoffs, including the low cutoffs, at once.

For general $b$, induct on $p$. Differentiation gives

$$\partial_b D_s^{(p)}=p(2b+2t+s)D_s^{(p-1)}.$$

If a sequence $d_s$ has all upper tails nonnegative and $w_s$ is nonnegative and nondecreasing, Abel summation gives

$$\sum_{s=S}^{2k} w_s d_s = w_S \sum_{s=S}^{2k} d_s + \sum_{j=S+1}^{2k} (w_j - w_{j-1}) \sum_{s=j}^{2k} d_s \geq 0. \tag{16}$$

Apply this with $w_s = 2b + 2t + s$ and the induction hypothesis. The derivative of each $p$-tail is nonnegative, and the value at $b = 0$ was proved above. This completes the induction. □

**Proposition 4.2** (Unbounded degree and geometric bias)**.** *For all integers $p, q \geq 0$ and $\rho \geq 1$, the target-minus-source sum of $(AB)^p(A+B)^q$, with the product weight $\rho^{i+j}$ on each rectangle, is nonnegative.*

*Proof.* On diagonal $s$ both rectangles have the same argument sum $A + B = 2b + 2t + s$. The difference is therefore

$$\sum_{s=0}^{2k} \rho^s (2b + 2t + s)^q D_s^{(p)}.$$

The prefactor is nonnegative and nondecreasing in $s$, including $q = 0$, so Lemma 4.1 and (16) apply. There is no bound on $p$ or $q$. □

## 5 From the middle strip to all heights

We now prove (2). Constants are harmless; suppose first that $h$ is nonconstant. Let $\sigma = h'/h$, which is nonincreasing by Section 2. The row score $(\log H_k)'$ is the $\rho^i h(y + ai)$-weighted mean of $\sigma(y + ai)$.

For $w = (u + v)/2$ and $\delta = (u - v)/2$, put

$$L(\delta) = \log H_{k-1}(w + \delta) + \log H_{k+1}(w - \delta), \quad -w \leq \delta \leq w.$$

Row concavity makes $L$ concave. At $\delta = 0$, adding the two largest arguments to the shorter prefix lowers its score, hence $L'(0) \geq 0$. If $w \geq a$, the shorter list at $\delta = a$ is $w + a, \ldots, w + ka$. The longer list is $w - a, \ldots, w + ka$; on their common part the weights differ by one fixed factor $\rho^2$, while the two additional arguments are smaller and have larger scores. Therefore $L'(a) \leq 0$. Concavity shows that the maximum over the feasible interval is attained in $[0, \min(a, w)]$. Thus it suffices to prove the inequality when $0 \leq u - v \leq 2a$.

For $a > 0$, divide all arguments by $a$ and replace $h(z)$ by $h(az)$, which preserves its zero sector. Write $u/a = b + 2t$, $v/a = b$, $b \geq 0$, $0 \leq t \leq 1$. Expand $h(A)h(B)$ by (4). Each term is nonnegative by Proposition 4.2, with precisely the rectangle weights obtained from multiplying the two prefixes. This proves the middle strip, hence all heights.

For $a = 0$, $H_k(y) = G_k(\rho)h(y)$, where $G_k(\rho) = \sum_{i=0}^{k} \rho^i$ and

$$G_k(\rho)^2 - G_{k-1}(\rho)G_{k+1}(\rho) = \rho^k.$$

Row log-concavity of $h$ gives the result directly. The same identity deals with a constant $h$, for every $a$. This completes the second assertion of Theorem 1.1.

## 6 Concave contact and integrable Jensen

Assume first that $h$ is nonconstant. For $q > 0$ define

$$M_k(q) = \sup_{y \geq 0} H_k(y) e^{-qy}, \qquad m_k(q) = \log M_k(q).$$

The supremum is finite and attained because $H_k$ is polynomial in $y$. The mixed inequality with independent second coordinates implies $M_{k-1}(q) M_{k+1}(q) \leq M_k(q)^2$, by taking separate maximizers. Thus $m_k(q)$ is a concave sequence. Let $\phi_q$ be its piecewise-linear interpolation and put

$$J(x, y) = \inf_{q > 0} \{\phi_q(x) + qy\}, \qquad x, y \geq 0. \tag{17}$$

Each function inside the infimum is concave, so their infimum is concave. Since $M_k(q) \geq H_k(0) \geq h(0)$, $J \geq \log h(0)$. Choosing one fixed $q > 0$ proves finiteness above. At any integer row and height $y \geq 0$, the positive supporting slope $q = (\log H_k)'(y)$ makes that height a maximizer in $M_k$, including the right supporting slope at zero. This proves $J(k, y) = \log H_k(y)$; the reverse inequality is immediate from the supremum definition. If $h$ is constant, use explicitly $J(x, y) = \log h(0) + \phi(x)$, where $\phi$ linearly interpolates $\log G_k(\rho)$. Equivalently exact contact follows by $q \downarrow 0$.

Finite concavity gives (3) for finite laws. For an arbitrary probability law, let $z = (\mathbb{E}K, \mathbb{E}Y)$ and let $F$ be the smallest coordinate face of the quadrant containing $z$. Every coordinate of the random point corresponding to a zero coordinate of $z$ vanishes almost surely, by nonnegativity. Hence the law is concentrated on $F$, and $z$ is in its relative interior. A finite concave function has a supporting affine majorant at every relative-interior point of its convex domain (apply the supporting hyperplane theorem to its hypograph). Thus there is an affine $L$ on $F$ with $L(z) = J(z)$ and $J \leq L$ on $F$. The assertion on a zero-dimensional face is immediate. The majorant also bounds all boundary points: concavity along a segment from $z$ to such a point and the interior supporting inequality imply that same bound there.

On the actual integer rows $J(K, Y) = \log H_K(Y)$ is measurable, since $K$ is integer-valued and each row is continuous. The bounds $\log h(0) \leq J(K, Y) \leq L(K, Y)$ and finite first moments imply absolute integrability. Taking expectations gives $\mathbb{E}J(K, Y) \leq L(z) = J(z)$. Integral $\mathbb{E}K$ supplies exact row contact at $z$, proving (3) without a countable-support or bounded-height restriction. This completes Theorem 1.1.

## 7 Sharpness and a geometric application

*Proof of Theorem 1.2.* For a wider angle choose $v > u > 0$ with $\arctan(v/u)$ inside it. The zeros of $h_L(t) = (t + Lu)^2 + (Lv)^2$ lie in that wider sector. At $K = 0$ the bias and step disappear, and direct calculation gives

$$h_L(0) h_L(2) - h_L(1)^2 = 2L^2(v^2 - u^2) - 4Lu - 1 > 0$$

for sufficiently large $L$. Equally weighted heights $0, 2$ have integral mean 1, contradicting logarithmic Jensen.

For $0 < \rho < 1$, use $h(t) = (t + 1)(t + 2)/2$, $K = 1$, and write $Z(y) = h(y) + \rho h(y + a)$. Exact expansion gives

$$Z(0) Z(2) - Z(1)^2 = \frac{\rho(1 - \rho)a^2 - 5\rho(1 + \rho)a - 6(1 + \rho)^2}{2}. \tag{18}$$

Its leading coefficient is positive. Hence a sufficiently large integer $a$ gives the claimed integral-height counterexample. □

**Corollary 7.1** (All-dimensional split bundles)**.** *Fix $m \geq 1$, $a \in \mathbb{Z}_{\geq 0}$ and $\rho \geq 1$. For integral $k, y \geq 0$ let*

$$Z_{m,a,\rho}(k,y) = \sum_{i=0}^{k} \rho^i \binom{y+ai+m}{m}.$$

*These functions satisfy logarithmic Jensen inequalities for arbitrary probability laws under the first-moment and integral-mean-prefix hypotheses of Theorem 1.1. For $\rho = 1$ they are section counts of the split bundle $\mathbb{P}_{\mathbb{P}^m}(\mathcal{O} \oplus \mathcal{O}(a))$, with the quotient convention for the tautological bundle, at $\mathcal{O}(k) \otimes \pi^*\mathcal{O}(y)$.*

*Proof.* The polynomial $h(t) = \binom{t+m}{m} = \prod_{j=1}^{m}(t+j)/m!$ has negative real zeros. The theorem applies, including the real height continuation. The projective-bundle pushforward is $\mathrm{Sym}^k(\mathcal{O} \oplus \mathcal{O}(a)) = \bigoplus_{i=0}^{k} \mathcal{O}(ai)$; taking sections gives the sum [9]. For $\rho \neq 1$ the formula is an explicitly weighted sum of these multiplicities, rather than an unweighted count. □

The twist $a$, dimension $m$ and bias are fixed in a Jensen law. When the mean height is also integral, both sides in the unweighted case refer to actual integral divisor parameters. The application is not a theorem for arbitrary toric varieties or arbitrary unimodular fibers.

The logarithm has a natural role here. For each fixed $\rho > 1$, no positive power can satisfy Jensen for all prefix lengths even with $a = 0$, $y = 0$ and constant input: for any $\alpha > 0$,

$$\frac{G_{k-1}(\rho)^\alpha + G_{k+1}(\rho)^\alpha}{2G_k(\rho)^\alpha} \longrightarrow \frac{\rho^{-\alpha} + \rho^\alpha}{2} > 1.$$

This elementary geometric-sequence observation does not obstruct our logarithmic conclusion. Nor is $J$ in (17) asserted to be the logarithm of an ordinary polynomial in real $x, y$.

## 8 Lattice overlap and reproducibility

For positive integer $c_j$ and unit step, the unweighted product prefix $\sum_{i=0}^{k} \prod_j (y + i + c_j)$ is a difference-bound lattice marginal. Take integer anchors $(x_0, x_1, x_2)$, states $v, w_j$ and inequalities

$$x_1 \leq v \leq x_0, \qquad v \leq w_j \leq x_2 + c_j - 1.$$

At anchors $(0, -k, y)$ the fibers have the indicated size, with $v = -i$. All constraints are fixed differences; the simultaneous translation symmetry gives the required $L$-convex model. The weight $\exp[\lambda(x_2 - v)]$ is logmodular and invariant under common translation. Speyer's weighted marginal Lemma 5.3 [8] therefore covers these integral unit-step cases for all real $\lambda$, after removal of the harmless factor $e^{\lambda y}$. This does not contradict (18), which uses larger steps.

Likewise, for integer positive $A, B$ and integer $p \geq 1$, $T_{p,r}(A, B)$ has a fixed difference-bound lift. Use anchors $(x_0, x_1, x_2) = (0, A - 1, 1 - B)$, states $I, J', X_l, Y_l$, and

$$I \geq x_0, \quad J' \leq x_0, \quad I - J' \leq r, \quad I \leq X_l \leq x_1, \quad x_2 \leq Y_l \leq J' \quad (1 \leq l \leq p).$$

With $J = -J'$, each feasible $(I, J)$ contributes $(A - I)^p(B - J)^p$, precisely the positive-part sum. Thus the integer-argument kernel already lies within the lattice marginal framework. The continuous real-argument assertion Theorem 3.1 requires additional work; the $p = 1$ example after Corollary 3.3 shows that integer marginal concavity alone does not ensure continuous log-concavity of this particular interpolation. Clearing rational denominators changes the step lattice and does not remove that distinction.

The public checker reconstructs the four complete chamber cases, the determinant identities, and the sector and sharpness polynomials using exact rational arithmetic and explicit error gates retained under Python optimization. It uses no external data, numerical eigenvalues, floating-point tolerance or random search. Its term counts are specified in Table 1; together with the formulas, this supplies a finite reproducible certificate for the algebraic portion of the proof. The analytic convolution, source enlargement, Abel induction, score reduction and contact construction are proved in the text, rather than inferred from checker output.

The assertions concern finite polynomials. The anisotropic kernel threshold in Corollary 3.3 is uniform over radii. Its fractional lift is a direct use of classical composition and marginalization. Further comparisons with spline and mixed discrete–continuous marginal inequalities may sharpen the relation to the existing literature.

**Disclosure.** AI-assisted tools were used for research, source comparison, exact algebra and internal verification. Distinct internal reconstructions checked the chamber identities and analytic composition. These are internal checks.

## References


[1] D. W. Walkup, *Pólya sequences, binomial convolution and the union of random sets*, J. Appl. Probab. **13** (1976), 76–85.

[2] S. G. Hoggar, *Chromatic polynomials and logarithmic concavity*, J. Combin. Theory Ser. B **16** (1974), 248–254.

[3] L. Gurvits, *On multivariate Newton-like inequalities*, in *Advances in Combinatorial Mathematics*, Springer, 2009, 61–78. arXiv:0812.3687.

[4] A. Marsiglietti and J. Melbourne, *Geometric and functional inequalities for log-concave probability sequences*, arXiv:2004.12005v4 (2023), especially Corollaries 3.3, 3.4 and 3.7. Full text.

[5] J. Melbourne and G. Palafox-Castillo, *A discrete complement of Lyapunov's inequality and its information theoretic consequences*, arXiv:2111.06997 (2021). arXiv:2111.06997.

[6] O. Johnson and C. Goldschmidt, *Preservation of log-concavity on summation*, ESAIM Probab. Stat. **10** (2006), 206–215. doi:10.1051/ps:2006008.

[7] A. Prékopa, *On logarithmic concave measures and functions*, Acta Sci. Math. (Szeged) **34** (1973), 335–343. Author copy.

[8] D. E. Speyer, *L-log-concavity and a proof of the conjecture of Lam, Postnikov and Pylyavskyy*, arXiv:2601.05007v2 (2026). arXiv:2601.05007.

[9] The Stacks Project Authors, *The Stacks Project*, Section 27.21, Example 27.21.3. Tag 01OA.

[10] S. Schubr, *Concavity of Lattice Counts on Toric Threefolds*, unpublished companion manuscript (2026).