A quantitative Chollet inequality for rank-four matrices of order seven

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Abstract

For every complex Hermitian positive semidefinite matrix A of order seven and rank at most four, we prove

$$\operatorname{per}(A \circ \overline{A}) \le \frac{192}{245}(\operatorname{per} A)^2.$$

The inequality is strict when all diagonal entries are positive. The main input is the bound $\operatorname{per}(A\circ\overline{A}) \le (3/5)(\operatorname{per} A)^2$ for every singular positive semidefinite matrix of order five. An exact rational principal-minor sum-of-squares certificate proves this singular-boundary inequality. A rank-preserving border comparison then gives the same constant in order six and the stated constant in order seven. The certificate and a standalone exact-integer verifier accompany the paper. Cauchy–Schwarz gives the corresponding two-factor inequalities when both factors have rank at most four.

Keywords: permanent; positive semidefinite matrix; Chollet inequality; rank four; sum of squares; computer-assisted proof.

1 Introduction and main results

For an $n \times n$ matrix $A = (a_{ij})$, write

$$\operatorname{per} A = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} a_{i,\sigma(i)}, \qquad P(A) = \operatorname{per} A, \qquad Q(A) = \operatorname{per}(A \circ \overline{A}),$$

where $\circ$ is the entrywise product and $\overline{A}$ is entrywise conjugation. All positive semidefinite matrices in this paper are complex Hermitian. Chollet's conjecture asks whether

$$\operatorname{per}(A \circ B) \le (\operatorname{per} A)(\operatorname{per} B) \quad \text{for all } A, B \ge 0$$

of the same order [1]. Its self-conjugate form is $Q(A) \le P(A)^2$; Cauchy–Schwarz on permutation monomials transfers a self-conjugate estimate to pairs in the same class.

Theorem 1.1. *Let $A \ge 0$ have rank at most four. Then*

$$Q(A) \le c_n P(A)^2, \qquad c_5 = c_6 = \frac{3}{5}, \qquad c_7 = \frac{192}{245}, \quad n \in \{5, 6, 7\}.$$

Each inequality is strict if every diagonal entry of A is positive. If a diagonal entry is zero, $P(A) = Q(A) = 0$.

Corollary 1.2. *Let $A, B \ge 0$ have order $n \in \{5, 6, 7\}$, with $\operatorname{rank} A \le 4$ and $\operatorname{rank} B \le 4$. With c_n as in Theorem 1.1,*

$$\operatorname{per}(A \circ B) \le c_n (\operatorname{per} A)(\operatorname{per} B).$$

The inequality is strict when both matrices have positive diagonal. If either matrix has a zero diagonal entry, both sides vanish.

The central step is a quantitative estimate on the singular boundary in order five. It is proved by an exact degree-ten polynomial identity modulo $\det A$, with principal-minor-weighted positive Gram forms. The bound $3/5$ compensates for the squared border loss $(8/7)^2$ in passing from order six to order seven. The estimates require neither tight-frame normalization nor entrywise or spectral bounds.

1.1 Related work

Li, Zhao and Luo develop a border-induction argument through order six [2]. We use their scalar and doubled occupation comparisons and give the complete portions needed here in Section 3. Lavi proves a quadratic Fourier-block inequality through order five and supplies a principal-minor sum-of-squares framework [3, 4]. The 38-block feature cone and symmetry organization used in this paper follow that framework. Our certificate has the different target $3P(A)^2 - 5Q(A)$ and adds a degree-five cycle-class multiplier of $\det A$ with unrestricted coefficients.

A direct dimension-counting consequence of Lavi's Theorem 4.1 and Fourier identity [3, Eq. (2.5)] is

$$Q(A) \le \frac{119}{120} P(A)^2 \quad \text{for singular } A \ge 0 \text{ of order five.}$$

Indeed, the sign block is $\det A = 0$, while the remaining block bounds have total dimension-squared weight 119, out of $5! = 120$. The stronger factor $3/5$ is sufficient for the order-seven transfer.

Ma's cited version treats rank at most four in orders at least ten and a tight-frame class in order nine; its remaining-scope discussion excludes rank-four orders seven and eight [5]. The quantitative order-seven research release [6] treats rank at most three, with factor $24/25$. Theorem 1.1 includes rank four and gives an explicit gap on that entire order-seven class.

2 Positivity and normalization

If $M_1, \dots, M_\ell \ge 0$ have order m, then the polynomial

$$\operatorname{per}(t_1 M_1 + \cdots + t_\ell M_\ell)$$

has nonnegative real coefficients. We recall the Gram-tensor argument because it also handles singular matrices.

Choose Gram vectors $u_{i,j}$ for M_j in mutually orthogonal Hilbert spaces and, for $t_j \ge 0$, put $v_i = \bigoplus_j \sqrt{t_j}\, u_{i,j}$. The unnormalized symmetrized tensor

$$T(v) = \sum_{\sigma \in \mathfrak{S}_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}$$

satisfies

$$\|T(v)\|^2 = m!\, \operatorname{per}(t_1 M_1 + \cdots + t_\ell M_\ell).$$

In its expansion, collect the complete sums having a fixed number of factors from each orthogonal Hilbert space. Distinct occupation profiles lie in orthogonal tensor subspaces. Thus each coefficient is a squared norm divided by $m!$, and is nonnegative. Equality on the positive parameter orthant identifies the polynomial coefficients. In particular $P(A) \ge 0$ for $A \ge 0$.

If $a_{ii} = 0$ in a positive semidefinite matrix, the principal two-by-two minors give $|a_{ij}|^2 \le a_{ii} a_{jj} = 0$. The ith row and column vanish, so $P(A) = Q(A) = 0$. For positive diagonal entries, a positive diagonal congruence reduces A to a correlation matrix. More precisely, for $D = \operatorname{diag}(d_1, \dots, d_n)$ with $d_i > 0$,

$$P(DAD) = \Big(\prod_i d_i^2\Big) P(A), \qquad Q(DAD) = \Big(\prod_i d_i^4\Big) Q(A). \tag{1}$$

Rank is unchanged. Consequently all the homogeneous inequalities in Theorem 1.1 are preserved by this normalization, without dividing by $P(A)$.

3 The rank-preserving border comparison

The following is the part of the border mechanism of [2] needed for the quantitative transfer.

Lemma 3.1. *Let $A \geq 0$ be a correlation matrix of order $n = m + 1$, written as*

$$A = \begin{pmatrix} B & x \\ x^* & 1 \end{pmatrix}, \qquad H = B - xx^*, \qquad R(t) = H + t\,xx^*.$$

Then $H \geq 0$ and, at $t = \sqrt{2}$,

$$Q(A) \leq Q(R(t)), \qquad P(R(t)) \leq b_m P(A), \qquad b_m = \max_{0 \leq k \leq m} \frac{2^{k/2}}{k+1}. \tag{2}$$

Moreover $\operatorname{rank} R(t) \leq \operatorname{rank} A$ *and* $R(t)_{ii} = 1 + (\sqrt{2} - 1)|x_i|^2 > 0$. *In particular $b_5 = 1$ and $b_6 = 8/7$.*

Proof. The Schur complement against the last diagonal entry gives $H \geq 0$, and the associated block congruence gives $\operatorname{rank} H = \operatorname{rank} A - 1$. Thus $\operatorname{rank} R(t) \leq \operatorname{rank} H + 1 = \operatorname{rank} A$.

Set $z_i = |x_i|^2$ and introduce the positive semidefinite matrices

$$U = H \circ \overline{H}, \quad V = H \circ \overline{xx^*} + \overline{H} \circ xx^*, \quad W = zz^T.$$

Their positivity follows from the Schur product theorem, or directly from Gram tensor products. Write

$$F(t) = P(H + t\,xx^*) = \sum_{k=0}^{m} \alpha_k t^k, \qquad \Gamma(a,b) = P(U + aV + bW) = \sum_{\substack{r,s \geq 0 \\ r+s \leq m}} \gamma_{rs} a^r b^s.$$

Section 2 gives $\alpha_k, \gamma_{rs} \geq 0$.

For an $m \times m$ matrix B, denote by $B(i \mid j)$ the matrix with row i and column j deleted. Expanding the permanent of A at its last row and column gives

$$P(A) = P(B) + \sum_{i,j} x_i \overline{x_j}\, P(B(i \mid j)) = F(1) + F'(1) = \sum_{k=0}^{m} (k+1)\alpha_k.$$

Indeed the sum is the directional derivative of $P(B)$ in the direction xx^*, and $B = H + xx^*$.

The doubled matrix $A \circ \overline{A}$ has border z and upper block $B \circ \overline{B} = U + V + W$. The same expansion, now with direction $W = zz^T$, gives

$$Q(A) = \Gamma(1,1) + \partial_b \Gamma(1,1) = \sum_{r,s} (s+1)\gamma_{rs}.$$

On the other hand,

$$P(R(t)) = F(t), \qquad Q(R(t)) = \Gamma(t, t^2).$$

At $t = \sqrt{2}$ we have $t^{r+2s} \geq 2^s \geq s + 1$, proving the first inequality in (2). The definition of b_m and nonnegative α_k prove the second.

For $0 \leq k \leq 5$, $2^{k/2} \leq k + 1$, with equality at $k = 0$. For $k = 6$, $2^{k/2} = 8$ and $k + 1 = 7$. This proves $b_5 = 1$ and $b_6 = 8/7$. Finally $H_{ii} = 1 - |x_i|^2$, giving the stated positive diagonal of $R(t)$. □

4 An exact inequality on singular matrices of order five

The certificate is finite exact mathematical data. We specify its polynomial interpretation before using its positivity.

Let $X = (x_{ij})_{1\le i,j\le 5}$ consist of 25 commuting formal variables. The involution $f \mapsto f^*$ sends x_{ij} to x_{ji} and fixes real coefficients. On Hermitian matrices it is complex conjugation. Define the formal polynomials

$$p(X) = \sum_{\sigma\in\mathfrak{S}_5} \prod_i x_{i,\sigma(i)}, \qquad q(X) = \sum_{\sigma\in\mathfrak{S}_5} \prod_i x_{i,\sigma(i)} x_{\sigma(i),i}.$$

They specialize to $P(A)$ and $Q(A)$. Let $G = \mathfrak{S}_5 \times \{1, {}^T\}$ act by simultaneous relabeling of rows and columns and by transpose. Its uniform Reynolds operator is

$$\mathcal{R}f = \frac{1}{240} \sum_{g\in G} gf.$$

For a partition λ of five, let

$$c_\lambda(X) = \sum_{\substack{\sigma\in\mathfrak{S}_5 \\ \mathrm{cyc}(\sigma)=\lambda}} \prod_i x_{i,\sigma(i)}.$$

There are seven such cycle types. Each c_λ is invariant under G.

Proposition 4.1 (Exact certificate)**.** *The accompanying file* `certificate/singular5_rho3_5_certificate.json` *specifies 38 monomial columns b_k, symmetric integer matrices N_k, products w_k of principal determinants, square integer matrices T_k, and integers h_λ, satisfying*

$$D(3p^2 - 5q) = \mathcal{R}\left(\sum_{k=1}^{38} w_k\, b_k^* N_k b_k\right) + (\det X) \sum_{\lambda\vdash 5} h_\lambda c_\lambda(X), \qquad D = 260963607394941665280. \quad (3)$$

For every k, the symmetric integer matrix $Z_k = T_k N_k T_k^T$ satisfies

$$(Z_k)_{ii} > \sum_{j\neq i} |(Z_k)_{ij}| \quad \textit{for every row } i. \quad (4)$$

The third block is factor-free: $w_3 = 1$, and its monomial column contains exactly the 119 nonidentity permutation monomials.

Exact verification. The accompanying standalone program `code/verify_certificate.py` expands (3) using integer arithmetic. A feature is encoded by its 25 nonnegative exponents in row-major order. The fields `factors`, `features` and `numerator` specify w_k, b_k and N_k; `congruence` specifies T_k. The seven entries of `multiplier_numerators`, indexed by `multiplier_classes`, specify h_λ.

The program constructs p^2 from all 120^2 ordered permutation pairs, constructs q from all 120 permutations, and expands each principal determinant. It reconstructs the Gram forms, applies the 240 group actions, and checks every coefficient in the resulting degree-ten identity after clearing the factor 240. The complete balanced support has 6,210 monomials, with row and column degree two; no additional monomial is permitted. It separately checks every integer inequality in (4) and the exact 119-feature set equality. Thus the checks establish a formal polynomial identity and exact positivity witnesses, without sampling matrices or accepting a floating-point tolerance.

The certificate was obtained using the principal-minor feature cone of [4], with the determinant multiplier in (3), semidefinite optimization and rational coefficient repair. Its validity depends on the displayed exact identity and integer witnesses, which the verifier reconstructs independently of the numerical discovery data. □

Remark 4.2 (Reproducibility). The certificate is an essential part of the proof. Its SHA-256 is

`e41c28e21094316d5813cac779e95b23`
`e64be66b4ddd8f98587b148faa8aa074.`

The hash identifies the supplied object; the coefficient and positivity checks prove the assertion. The verifier uses the Python standard library only. From the package root, run `python code/verify_certificate.py`; the same checks remain active with `python -O`. No optimizer, numerical solution, external package, or predecessor certificate is needed for verification.

Lemma 4.3. *For every singular $A \geq 0$ of order five,*

$$Q(A) \leq \frac{3}{5} P(A)^2.$$

The inequality is strict for positive diagonal. A zero diagonal gives $P(A) = Q(A) = 0$.

Proof. A real symmetric matrix Z satisfying (4) is positive definite, also on complex vectors, because

$$u^* Z u \geq \sum_i \left(Z_{ii} - \sum_{j \neq i} |Z_{ij}| \right) |u_i|^2 > 0 \quad (u \neq 0).$$

Since $Z_k = T_k N_k T_k^T$ is positive definite and T_k is square, T_k is invertible; otherwise T_k^T would have a nonzero kernel. It follows that every N_k is positive definite.

All principal determinants of A are nonnegative, so $w_k(A) \geq 0$ and $w_k(A) b_k(A)^* N_k b_k(A) \geq 0$. The group transformations preserve Hermitian positive semidefiniteness, so their average is nonnegative. Singularity makes the determinant term in (3) vanish. This proves the weak inequality.

Suppose the diagonal is positive. A singular diagonal matrix with positive diagonal is impossible, so some $a_{ij} \neq 0$ with $i \neq j$. The transposition (ij) contributes the strictly positive feature

$$|a_{ij}|^2 \prod_{\ell \notin \{i,j\}} a_{\ell\ell}$$

to $b_3(A)$. Thus $b_3(A) \neq 0$ and its factor-free positive-definite Gram form is strictly positive. The Reynolds average includes this term with positive weight and all other terms are nonnegative. Hence $3P(A)^2 - 5Q(A) > 0$. The zero-diagonal case was established in Section 2. □

5 Proof of the main results

Proof of Theorem 1.1. Zero diagonal entries give the stated equality. Otherwise (1) reduces the proof to correlation matrices.

In order five, rank $A \leq 4$ gives $\det A = 0$, so Lemma 4.3 applies. In order six, form $R = R(\sqrt{2})$ as in Lemma 3.1. It has order five, rank at most four, and positive diagonal. Consequently

$$Q(A) \leq Q(R) < \frac{3}{5} P(R)^2 \leq \frac{3}{5} P(A)^2.$$

Nonnegativity of PSD permanents justifies squaring the scalar comparison.

In order seven, R has order six, rank at most four, and positive diagonal. The order-six result just proved and $b_6 = 8/7$ give

$$Q(A) \leq Q(R) < \frac{3}{5} P(R)^2 \leq \frac{3}{5} \left(\frac{8}{7} \right)^2 P(A)^2 = \frac{192}{245} P(A)^2.$$

This includes all rank-deficient cases, without an invertibility assumption on B or R. □

Proof of Corollary 1.2. For $a_\sigma = \prod_i a_{i,\sigma(i)}$ and $b_\sigma = \prod_i b_{i,\sigma(i)}$, Cauchy–Schwarz gives

$$|\operatorname{per}(A \circ B)| = \left| \sum_\sigma a_\sigma b_\sigma \right| \leq \left(\sum_\sigma |a_\sigma|^2 \right)^{1/2} \left(\sum_\sigma |b_\sigma|^2 \right)^{1/2} = \sqrt{Q(A) Q(B)}.$$

Theorem 1.1 proves the claim and its strict form. The Schur product theorem makes $\operatorname{per}(A \circ B) \geq 0$. A zero diagonal entry in either factor gives a zero row in $A \circ B$, and the asserted equality follows. □

Data and code

The accompanying package contains the exact certificate, the standalone verifier, this paper's complete LaTeX source, and reader instructions. The proof uses no numerical search output. The format and coefficient checks are described in Proposition 4.1.

AI assistance

AI tools assisted drafting and verification; the author is responsible for the content.

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